INTRODUCTION TO SPACE TECHNOLOGY (R15A2107)

COURSE FILE

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UNIT-I

Fundamentals of Rocket Propulsion

1.1 SPACE ENVIRONMENT AND SPACE MISSIONS

In the design of mission, spacecraft and launch systems the influence of prevailing natural environment need to be considered. We discussed some of these in connection with the ascent phase of a launch vehicle through the atmosphere. In addition to atmosphere, solar radiation field, meteoroids and earth's magnetic field dictate to an extent the mission and the spacecraft design. In the recent years in addition to the above mentioned natural factors, manmade factors such as increasing presence of space debris need also to be taken into account. In this section we briefly discuss the characteristic features of some of the natural factors.

1.1.1 EARTH'S ATMOSPHERE

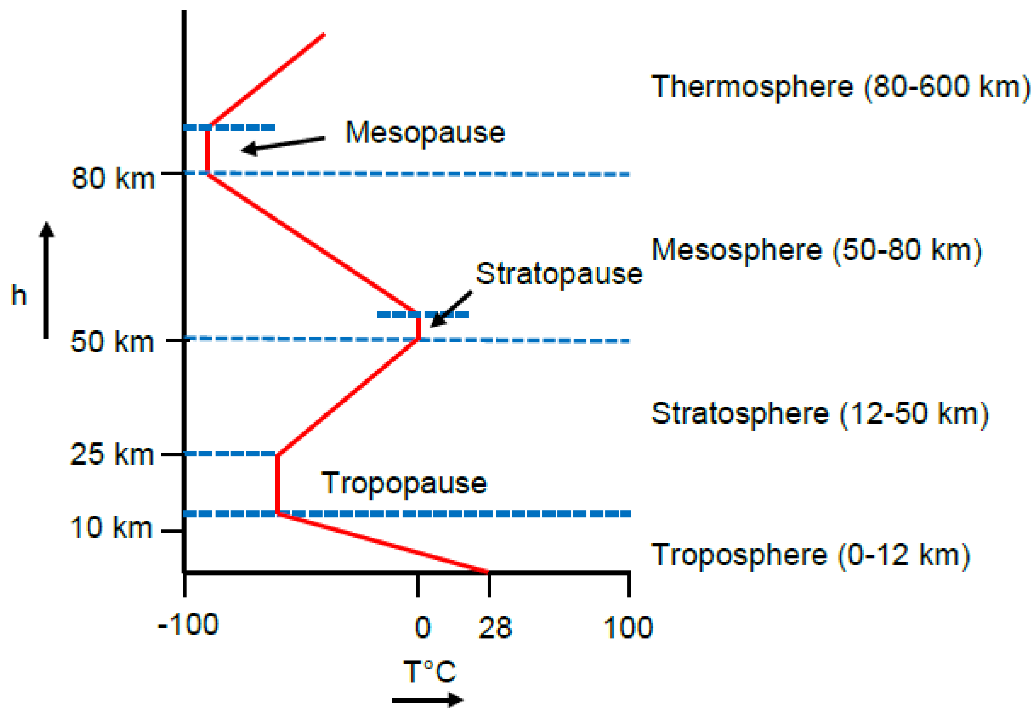

Figure 1

Earth's atmosphere consists of air which composes (by volume) 78% of nitrogen, 21% oxygen and argon, carbon-dioxide and other gases constituting the remaining 1%. The atmosphere is classified into different layers based on the change in temperature.

The first layer, Troposphere ranges from sea-level to 12 km and the temperature and pressure drop steadily in this layer. This layer consists of 75% of the total atmospheric mass. There is region with a width of 15 km where the temperature reaches its minimum named Tropopause. Then the second layer is stratosphere where the temperature increases due the absorption of the sun light (mainly ultra violet radiation) by the ozone gas present here. This layer extends up to 50 km altitude. Then over a small layer of Stratopause (around 10 km), the temperature reaches a local maximum. Mesosphere is the third layer extending up to 80 km where temperature decreases. The air is thinner here but the meteorites from space end up burning in this layer. Next, there is a Mesopause with a constant minimum temperature. The fourth layer Thermosphere extends from 80-600 km consists of very few molecules of gases and the temperature increases in this layer. These molecules absorb the solar radiation to reach very high temperatures (over 1000° C). But one would feel this layer to be colder since very few molecules collide with skin and would transfer very less heat.

1.1.2 SOLAR RADIATION

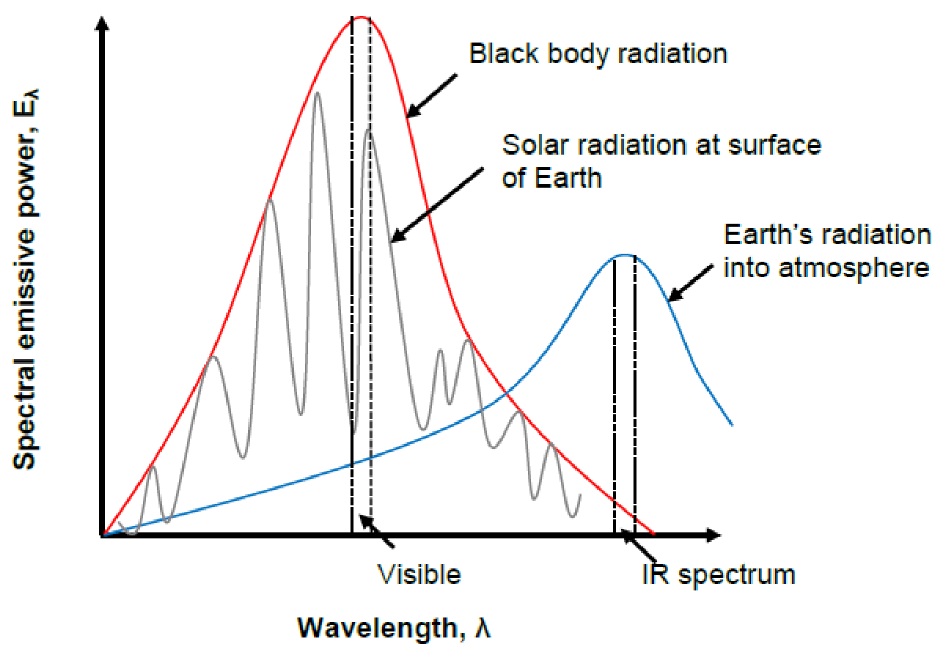

Figure 2

The electromagnetic spectrum of sun's radiation closely resembles that of a black body at a temperature of 5800 K. Sun emits its maximum spectral energy over the visible wavelength (0.4-0.7 µm). The discontinuous or irregular variation of the spectral emissive power is due to absorption of solar radiation in spectrum where atmospheric gases radiatively participate in absorbing the incoming solar radiation.

The Earth's radiation back into atmosphere resembles a black body radiation at lower temperature such 300 K. One can observe that the maximum spectral energy is radiated from Earth is in the infra-red spectrum of wavelength (1-100 µm).

1.1.3 EARTH'S MAGNETIC FIELD

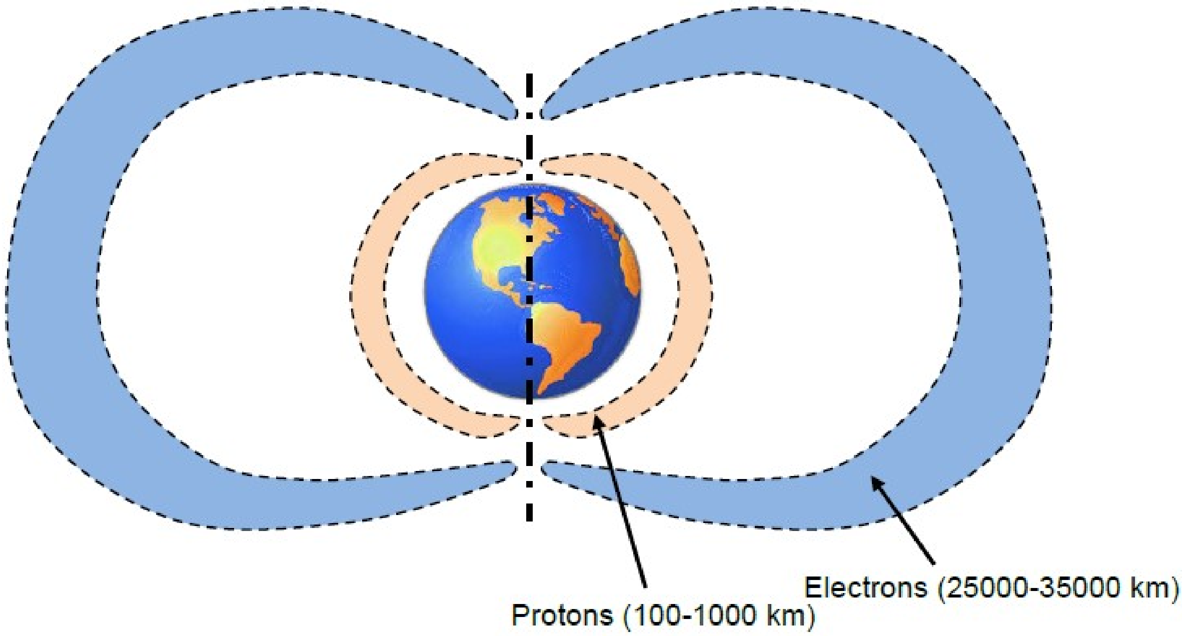

Figure 3

Earth's magnetic field results from constant convection of Earth's molten core. The magnetic axis is titled by 11° to its rotational axis and hence magnetic and geometric north and south poles do not coincide with each other. The magnetic field protects the Earth from high energy cosmic rays emanating from the Sun. These high energy particles (protons and electrons) get trapped by Earth's magnetic field forming two distinct regions of charged particle layers termed as Van Allen radiation belts.

The inner region consists of protons and the outer region consists of electrons. As these charged particle belts are spread across the outer layers of atmosphere, they pose radiation threat to the satellites orbiting Earth. For example, the Molniya orbits which have their orbits which varying from low to very high altitude would be crossing the radiation belts. In general, the satellites use radiative shields and their orbits are designed so that only minimum possible time is spent in passing through the Van Allen belts.

1.2 SPACE MISSIONS

The space mission may be broadly classified as near earth missions which primarily comprises of earth-satellite system and deep space missions that are targeted beyond earth.

1.2.1 NEAR EARTH MISSIONS SPACE CRAFTS AND THEIR AND ORBITS

Some of the common near earth space missions and their orbits are describe below:

(1) Geostationary orbit:

The orbit located on the equatorial plane such that the time period of rotation of satellite is same as the time period of rotation of earth about its axis. The projection on to surface of earth is a steady point.

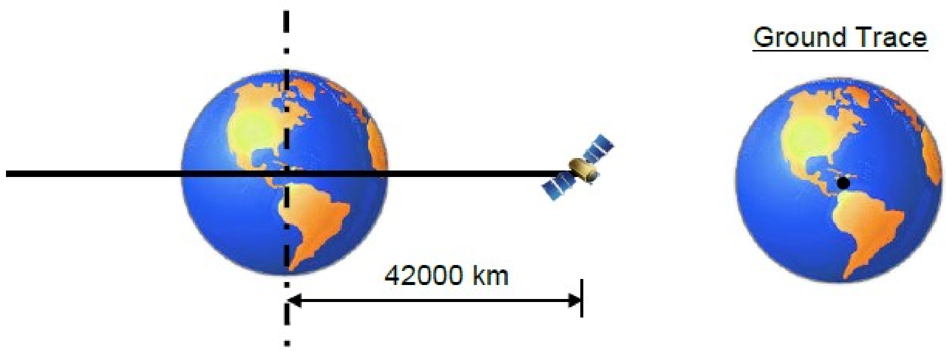

Figure 4 Geostationary orbit

(2) Geosynchronous orbit:

Orbits inclined to the equatorial plane with the time period of rotation of satellite is same as the time period of rotation of earth about its axis. The projection on to surface of earth varies with time and the projection point traces a figure of ‘8’ on the surface of earth.

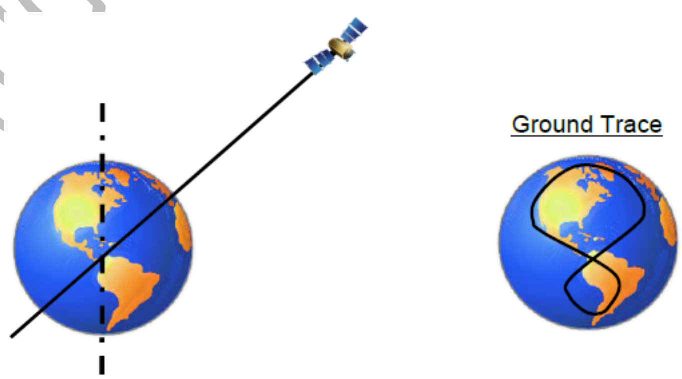

Figure 5 Elliptical inclined orbits (e.g. 24 hour Tundra orbit)

(3) Other inclined orbits:

24 hour (Tundra) and 12 hour (Molniya) orbits are highly elliptical orbits which are used by Russia located far north where the geostationary satellite are not visible (usually above 55° N latitude). These satellites move slowly over the country of interest (near apogee) and fast when away from the country (near pergee)

(4) Polar orbits:

The satellites that pass over or close to the north and south poles (see figure 6) of the earth are said to be orbiting in polar orbits. Polar orbits are generally circular and thus move at a constant altitude above the ground which may be between 200 km-1000 km. The earth below rotates while the satellite orbits pole to pole. This orbit is ideal for remote sensing with high resolution images, mapping and meteorology missions.

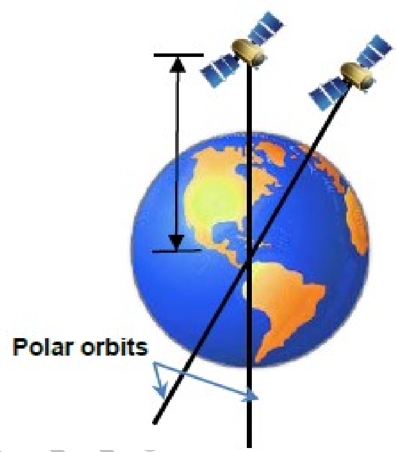

Figure 6 Polar Orbit

(5) Sun synchronous orbit

The polar orbits determined for time synchronizing with sun to get some illumination are called the Sun synchronous orbits.

1.3 DEEP SPACE MISSIONS AND THEIR SPACECRAFT

The deep space missions maybe classified based on the target as:

(1) Inner Solar system targets – Mercury, Venus, Mars, Moon

(2) Outer Solar system and beyond targets – Jupiter, Saturn, Uranus

(3) Solar orbitals – Asteroids and Comets

The missions (and also the spacecraft for these missions) may also be classified based on the objectives as

1) *Observatory mission*

These include installing space observing telescopes located at very high altitudes. For example Spitzer space telescope launched in 2003 was placed in heliocentric orbit with time period of 1 year. Chandra-X-ray telescope launched in 1999 with life expectancy of about 15 year's moves in geocentric orbit with apogee at 133000km and perigee at 16000 km.

2) *Engineering /Technology demonstration mission*

These are launched to prove certain concepts or technology for future space applications. The example include Deep space-1 launched in 1998 to Asteroid Braille tested somewhere about 12 technologies including electric ion propulsion for such missions.

3) *Flyby mission*

In these missions the spacecraft makes observation from a solar orbit or escape trajectory as it passes by the object target. Voyager 1 & 2 flew by the large planets out of solar system. New Horizons launched in 2006 is schedule to perform flyby of Pluto sometime in 2015.

4) *Target object orbit mission*

These missions are designed to take spacecraft to a target object and upon arrival orbit about the object. The examples of such mission include Galileo mission to orbit Jupiter in 1995, Cassini mission to orbit Saturn 2004, Chandrayaan mission to orbit Moon in 2008.

5) *Target object atmospheric entry or landing mission*

The atmospheric spacecraft penetrates and study the atmosphere of the target (e.g. Huygens, 2004, to moon of Saturn). The lander spacecraft may land gently on the surface of target planet (e.g. Mars Pathfinder Sojourner, 1997) impact at high velocity.

1.4 ROCKET EQUATION, ROCKET PROPULSION-TYPES:

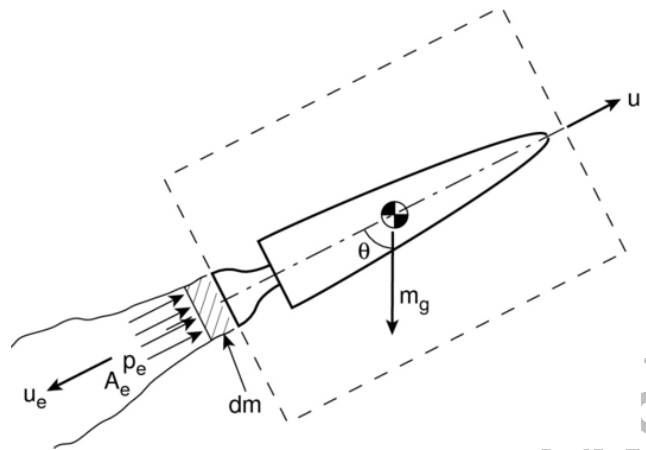

Figure 7 Schematic for application of the momentum theorem.

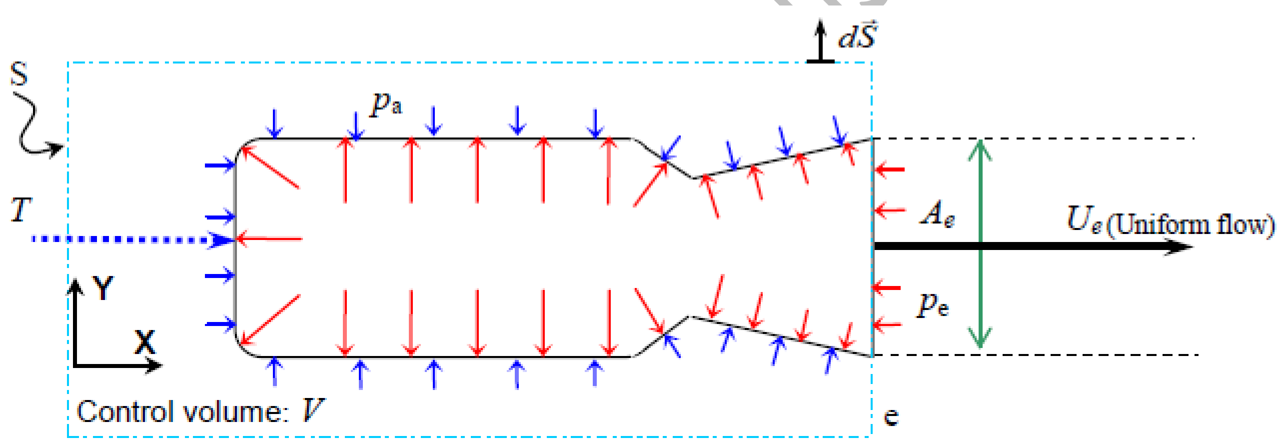

Figure 8 Schematic showing control volume enclosing a stationary rocket motor/engine.

1.4.1 BASIC TERMS:

Thrust (T):

$$T = \dot{m}_e U_e + A_e(p_e - p_a) = \dot{m}_e U_{eq}$$

$$\text{where } U_{eq} = U_e + \left(\frac{p_e - p_a}{\dot{m}}\right) A_e$$

Total impulse (I):

The total impulse I is the thrust force T (which can vary with time) integrated over the burning time t.

$$I = \int T\,dt = \int \dot{m}_e U_{eq}\,dt = m_p U_{eq} \qquad units: \left(kg\,\frac{m}{s}\right)$$

where m_p = total mass of propellant expelled.

Specific impulse (I_{sp}):

The specific impulse I_{SP} is the total impulse per unit weight of propellant.

$$I_{sp} = \frac{I}{m_p g_e} = \frac{U_{eq}}{g_e} \qquad units: \left(\frac{m/s}{m/s^2} \equiv s\right)$$

Here g_e is the acceleration due to gravity at the earth's surface. Note that the choice of g_e is arbitrary. The advantage is that in all common systems (fps, cgs, SI etc.) the unit of specific impulse (I_{sp}) is the same '*seconds*'.

1.4.2 ROCKET EQUATION

There are several ways to do this through applying conservation of momentum. Here we will apply the momentum theorem differentially by considering a small mass *dm*, expelled from the rocket during time *dt*, Figure 7

The initial momentum of the mass in the control volume (the vehicle) is $m_v u$. The final momentum of mass in the control volume (the vehicle and the mass expelled, *dm* is

$$(m_v - dm)(u + du) + dm(u - u_e) = m_v u + m_v du - udm - dudm + udm - u_e dm. \qquad (1)$$

The change in momentum during the interval dt is

$$\text{change in momentum} = \text{momentum}_{\text{final}} - \text{momentum}_{\text{initial}} = m_v du - u_e dm, \qquad (2)$$

Since $dudm$ is a higher order term.

Now consider the forces acting on the system which is composed of the masses m (the rocket), and dm (the small amount of propellant expelled from the rocket during time dt

$$\sum F = (p_e - p_0)A_e - D - mg\cos\theta. \tag{3}$$

Applying conservation of momentum, the resulting impulse, , must balance the change in momentum of the system:

$$m_v du - dm u_e = [(p_e - p_0)A_e - D - m_v g\cos\theta]dt. \tag{4}$$

Then since $dm = \dot{m}dt = -\frac{dm_v}{dt}dt,$ (5)

Where mdot is the propellant mass flow rate, we have

$$m_v du = [(p_e - p_0)A_e + \dot{m}u_e - D - m_v g\cos\theta]dt, \tag{6}$$

or, for $p_e = p_0$

$$du = -\frac{u_e dm_v}{m_v} - \frac{D}{m_v}dt - g\cos\theta dt. \tag{7}$$

Equation (7) is known as The Rocket Equation. It can be integrated as a function of time to determine the velocity of the rocket.

If we set $u_e = \text{constant}$, assume that at $t = 0$, $u = 0$, neglect drag, and set $\theta = 0$, then we can simplify the rocket equation to

$$du = -u_e\frac{dm_v}{m_v} - gdt, \tag{8}$$

Which can be integrated to give

$$u = -u_e \ln\left(\frac{m_v}{m_{v0}}\right) - gt, \tag{9}$$

where m_{v0} is the initial mass of the rocket. We can also write this result as

$$u = g\left[\text{Isp}\ln\left(\frac{m_{v0}}{m_v}\right) - t\right] \tag{10}$$

Assuming the rate of fuel consumption is constant, the mass of the rocket varies over time as

$$m_v(t) = m_{v_0} - (m_{v_0} - m_{v,\text{final}})\frac{t}{t_b}, \tag{11}$$

where t_b is the time at which all of the propellant is used. This expression can be substituted into the equation for velocity and then integrated to find the height at the end of burnout:

$$h_b = \int_0^{t_b} u dt, \tag{12}$$

Which for a single stage sounding rocket with no drag and constant gravity yields.

1.4.3 ROCKET PROPULSION

1.4.3.1 TYPES

1.4.3.1.1 CHEMICAL PROPULSION

Chemical propulsion is propulsion in which the thrust is provided by the product of a chemical reaction, usually burning (or oxidizing) a fuel. A chemical reaction combines two or more kinds of chemicals and makes a different chemical as a product. A commonly used reaction is combining hydrogen with oxygen to make water.

Generally, the reaction also releases heat which will warm up the product. Since when you heat a substance it expands (try this with a crayon and a magnifying glass), the heat produced by the chemical reaction heats up the product, making it expand. As it expands, it gets too big for the reaction chamber and pushes out the back of the rocket. This provides thrust for the rocket.

There are several types of chemical rocket propulsion systems and same are described in the following table

Type	Uses	Advantages	Disadvantages
Solid fuel chemical propulsion	main booster	simple, reliable, few moving parts, lots of thrust	not restartable
Liquid fuel chemical propulsion	main booster, small control	restartable, controllable, lots of thrust	complex
Hybrid fuel chemical propulsion	Combination of solid +liquid fuels	restartable, controllable, lots of thrust	complex
Cold-gas chemical propulsion	small control	restartable, controllable	low thrust
Other types	in space booster	restartable, controllable, high specific impulse	complex

SOLID FUEL CHEMICAL PROPULSION

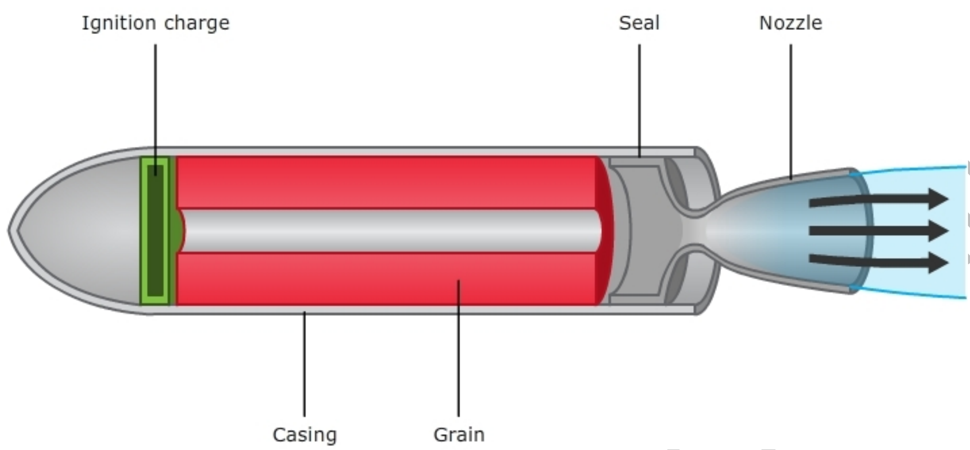

Figure 9 Schematic of Solid Fuel Chemical Propulsion

In a solid rocket, the fuel and oxidizer are mixed together into a solid propellant which is packed into a solid cylinder. A hole through the cylinder serves as a combustion chamber. When the mixture is ignited, combustion takes place on the surface of the propellant. A flame front is generated which burns into the mixture. The combustion produces great amounts of exhaust gas at high temperature and pressure. The amount of exhaust gas that is produced depends on the area of the flame front and engine designers use a variety of hole shapes to control the change in thrust for a particular engine. The hot exhaust gas is passed through a nozzle which accelerates the flow. Thrust is then produced according to Newton's third law of motion.

The amount of thrust produced by the rocket depends on the design of the nozzle. The smallest cross-sectional area of the nozzle is called the throat of the nozzle. The hot exhaust flow is choked at the throat, which means that the Mach number is equal to 1.0 in the throat and the mass flow rate m dot is determined by the throat area. The area ratio from the throat to the exit A_e sets the exit velocity V_e and the exit pressure p_e. The exit pressure is only equal to free stream pressure at some design condition. We must, therefore, use the longer version of the generalized thrust equation to describe the thrust of the system. If the free stream pressure is given by p0, the thrust F equation becomes

$$F = m\ dot * Ve + (p_e - p_0) * A_e$$

Note: there is no free stream mass times free stream velocity term in the thrust equation because no external air is brought on board. Since the oxidizer is mixed into the propellant, solid rockets can generate thrust in a vacuum where there is no other source of oxygen. That's why a rocket will work in space.

Solid motor is used mainly as a booster for launch vehicles. Solid motors are almost never used in space because they are not controllable. The boosters are lit and then they fire until all the propellant has burned. Their main benefits are simplicity, a shelf life which can extend to years as in the case of missiles, and high reliability.

In general, Space Shuttle has two solid rocket boosters (SRBs). These are the two big white rocket sections on the side of the Space Shuttle that produce the visible flames and smoke. The SRBs are the largest solid fuel engines ever used in a launch. Each SRB burns nearly 4000 kg of propellant each second and ejects the resulting hot gases to produce a thrust of 12.5 mega newton (MN). Compare this with much smaller engines for model rockets that can be made to produce as little as 2 newton (N) of thrust.

LIQUID FUEL CHEMICAL PROPULSION

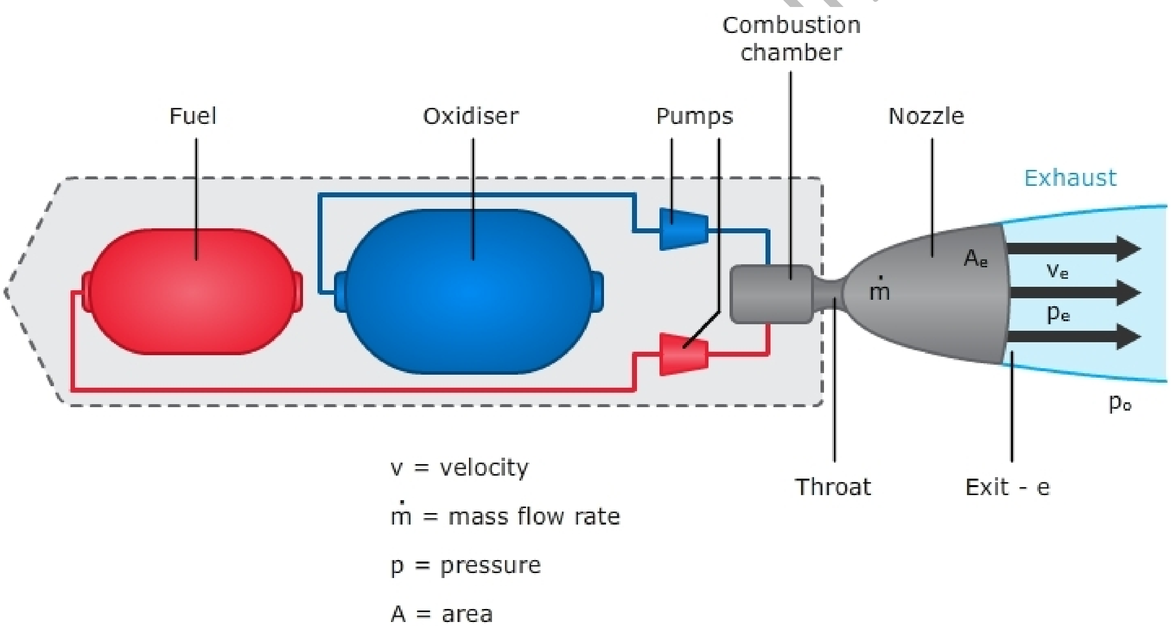

Figure 10 .Schematic of Liquid Fuel Chemical Propulsion

Liquid rocket engines are used on the Space Shuttle to place humans in orbit, on many un-manned missiles to place satellites in orbit, and on several high speed research aircraft following World War II. In a liquid rocket, stored fuel and stored oxidizer are pumped into a combustion chamber where they are mixed and burned. The combustion produces great amounts of exhaust gas at high temperature and pressure. The hot exhaust is passed through a nozzle which accelerates the flow. Thrust is produced according to Newton's third law of motion.

The amount of thrust produced by the rocket depends on the mass flow rate through the engine, the exit velocity of the exhaust, and the pressure at the nozzle exit. All of these variables depend on the design of the nozzle. The smallest cross-sectional area of the nozzle is called the throat of the nozzle. The hot exhaust flow is choked at the throat, which means that the Mach number is equal to 1.0 in the throat and the mass flow rate m dot is determined by the throat area. The area ratio from the throat to the exit A_e sets the exit velocity V_e and the exit pressure p_e. The exit pressure is only equal to free stream pressure at some design condition. We must, therefore, use the longer version of the generalized thrust equation to describe the thrust of the system. If the free stream pressure is given by p_0, the thrust F equation becomes:

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Liquid motors come in many shapes and sizes: Most of them are controllable (can be throttled up and down), restartable, are often used as control and maneuvering thrusters. Liquid thrusters can be broken into three main types:

- Monopropellant
- Bipropellant
- Cryogenic thrusters.

Monopropellants

Monopropellant hydrazine thrusters have typical performance characteristics as follows:

- Thrust range: 0.025-125 lbf
- Isp range: 225-239 lbf-sec/lbm
- Restart capability: 750,000 starts at 50 lbf
- Pressure operating range: 350 psia blowdown at 100 psia
- Radiative thermal control

Bipropellants

The bipropellant chemical propulsion system MON/MMH has typical performance characteristics as follows:

Thrust range: 0.4-5 lbf

Isp range: 250-295 lbf-sec/lbm

Restart capability: multiple

Pressure operating range: 350 psia blowdown at 100 psia

Radiative thermal control

In this same thrust class, an innovative bipropellant thruster, secondary combustion augmented thruster (SCAT), has been flight qualified and flown. This thruster operates in the bipropellant mode on MON/N2H4 until the oxidizer is expended and then operates as a monopropellant thruster until all the fuel is expended. Northrop Grumman is the sole supplier for this engine.

The bipropellant chemical propulsion systems MON/MMH and MON/N_2H_4, high thrust, are used in liquid apogee engines. They have the following typical performance characteristics:

Thrust range: 100-110 lbf

Isp range: 305-326 lbf-sec/lbm

Restart capability: multiple

Engine inlet operating pressure: 250 psia

Radiative/film thermal control

MON/MMH liquid apogee engines are typically used in combination with the low-thrust MON/MMH thrusters used for on-orbit propulsive functions. MON/N_2H_4 liquid apogee engines are advantageous for spacecraft propulsion systems that use monopropellant hydrazine or electro thermal hydrazine or hydrazine arc jet thrusters for on-orbit propulsive functions.

In general, three main engines on the tail of the Space Shuttle orbiter are liquid fuel rocket engines. The external tank (ET) is the big orange tank and contains two separate storage tanks – one containing liquid hydrogen and one containing liquid oxygen. The hydrogen and oxygen are pumped to the three main engines. They are sprayed into a combustion chamber where the hydrogen reacts with the oxygen to form gaseous water. It is the high-speed ejection of this gaseous water that produces the thrust. Each main engine produces a thrust of 1.8 MN (1.8 million N). It does this by reacting 1340 liters of propellant each second and ejecting the gaseous water at a speed of 3560 m/s (12 800 km/h).

Cryogenic systems use liquefied gases such as LiH and LOX (liquid hydrogen and liquid oxygen). Cryogenic means super-cooled. You would have to super-cool hydrogen and oxygen to make them liquids. With each step from monopropellant to bipropellant to cryogenic the thruster complexity goes up but the performance also goes up.

HYBRID FUEL CHEMICAL PROPULSION

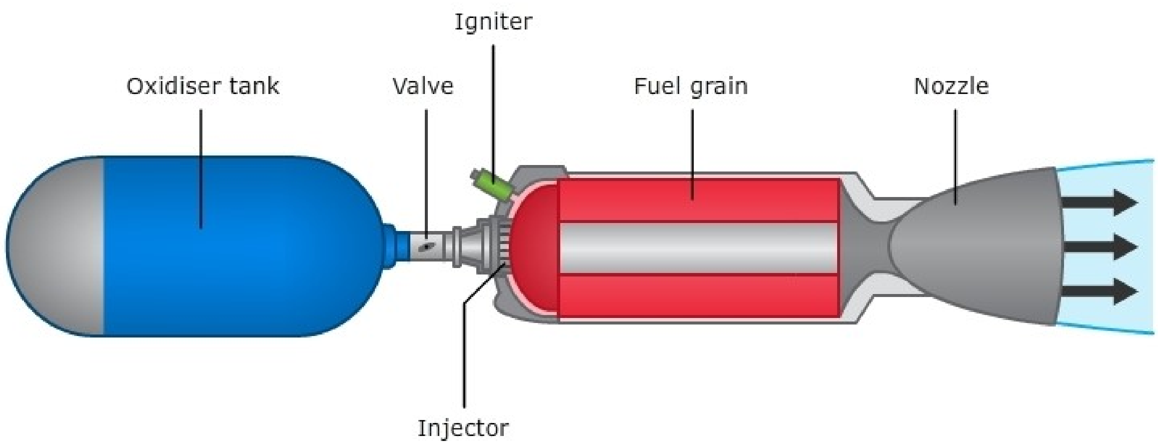


Figure 11 .Schematic of hybrid Fuel Chemical Propulsion

A hybrid system, as its name suggests, consists of one solid propellant and one liquid propellant. The fuel can be either the liquid or the solid and the same goes for the oxidizer. However, useful oxidizers tend to be liquids and so the typical configuration of a hybrid rocket consists of a liquid oxidizer reacted with a solid fuel. Figure 11 displayed above illustrates a typical configuration for a hybrid rocket. In this diagram, the fuel is contained within the combustion chamber in the form of a ported cylinder. The oxidizer is stored separately in a tank and when thrust is desired the valve is opened and vaporized oxidizer flows down the port where combustion takes place.

A turbulent diffusion flame is established over the fuel surface as shown in the figure above. Heat transfer from the flame vaporizes the fuel sustaining combustion and the fuel surface regresses in the radial direction as it is consumed. Combustion occurs in the port near the surface of the fuel and the fuel regresses in the radial direction as it is consumed. Hybrids combine some of the advantages of liquids and solids and also exhibit unique advantages. The most important is probably the inherent safety associated with storing the fuel in the solid phase. A general summary of the other advantages is given in the table below:

	Solid Rockets	Liquid Rockets
Simplicity	Chemically Simpler Tolerant of processing errors	Mechanically Simpler Tolerant of fabrication errors
Safety	Reduced chemical explosion hazard	Reduced fire hazard Less prone to hard-starts
Operability	Throttling, Start/Stop/Restart Capability	Operation requires only a single liquid
Performance	Higher Specific Impulse (Isp)	Higher fuel density Easy inclusion of high-energy additives (Al, Be, etc.)
Environmental	No perchlorates required Non-toxic exhaust products	Solid fuel presents reduced contamination hazard
Cost	Reduced development costs Reduced recurring costs	

Cold-gas motors have controllability similar to liquids but are the simpler and lighter. They are basically a high pressure tank with switches which flip between the open and shut state. They function a little like spray paint, with the contents under pressure inside, and when the valve is opened, they stream out.

1.4.3.2. OTHER TYPES

Electric Propulsion

The expanding range of spacecraft sizes and the changes in the commercial spacecraft industry environment have been presenting new challenges to the chemical propulsion community. There has been a clear need for higher performance propellants and/or thrusters. The advent of power-rich spacecraft architectures provides opportunities to take advantage of various propulsion options that can provide both high power and high Isp. Reducing an onboard propulsion system's wet mass requirement can either decrease total spacecraft mass or increase payload capacity. In addition, greater demands can be placed on the propulsion system, including more calls for repositioning or longer duration orbit maintenance, increasing useful life. Another option enabled by a reduced wet mass might be a stepdown to a lower-weight-class launch vehicle. These performance enhancements, which are of great interest to commercial satellite owners, are also desirable for military satellites. The propulsion industry has accepted these challenges and is transitioning to electric propulsion.

ELECTROTHERMAL THRUSTERS

Starting with the implementation in the 1980s of EHTs on INTELSAT V and RCA Astro Electronics communication satellites, a 30 percent improvement in performance, from an Isp of 225 sec to an Isp of 295 sec, was achieved with this thruster type, which electrically heats the decomposition products of catalytically decomposed monopropellant hydrazine to higher chamber temperatures. Without the complexity of carrying an oxidizer on board, this Isp is competitive with low-thrust bipropellant systems (0.4 to 5 lbf), which provide an Isp of 295 sec. TRW Space and Communications (now Northrop Grumman), Redondo Beach, built the EHTs for INTELSAT V. At present it does not manufacture this thruster type.

ARCJETS

In the early 1990s, Lockheed Martin utilized a new thruster, the hydrazine arc jet for North-South station keeping (NSSK) on its geostationary orbit satellites. The early R&D (through preflight qualification) on the hydrazine arc jet was done at NASA Glenn Research Center. The current production arc jet thrusters are manufactured by Aerojet Redmond. The Lockheed Martin series 7000 satellites use the Aerojet MR 509 hydrazine arc jet system (1.8-kW power level, Isp of 502 sec). The arc jet continues to evolve with the latest Lockheed satellite bus, the A2100 satellites, which utilize the MR-510 arc jet system (2.2-kW, 582-sec nominal Isp thrusters) for NSSK. Again, the arc jet thruster takes advantage of the higher satellite power available to substantially increase the performance over catalytic hydrazine (Isp = 225 sec to Isp = 570 to 600 sec).

ION THRUSTER SYSTEMS

The thruster consists of a discharge hollow cathode, three-ring magnetic cusp confinement, a three-grid accelerator, and neutralizer hollow cathode. The three-grid accelerator used in the 25-cm thruster utilizes shaped molybdenum grids with approximately 11,000 apertures to produce the high perveance (72 pervs at full power) xenon ion beam. The XIPS 25-cm ion thrusters and the associated power supplies operate in two modes: 2.2 kW for typical on-orbit functions and 4.4 kW for raising the orbit. The high-power mode utilizes about 4.5 kW of bus power to produce a 1.2-kV, 3-Å ion beam. The thruster in this mode produces 165 mN thrust at an Isp of about 3,500 seconds. The high-power mode is used exclusively for the orbit insertion phase, which greatly reduces the amount of chemical propellant carried by the spacecraft for this task. Nearly continuous operation in the high-power mode for 500 to 1,000 hours is required, depending on the launch vehicle and satellite weight.

A low-power mode, in which the thruster consumes about 2.2 kW of bus power, is used for the station-keeping function. In the low-power mode, the beam acceleration voltage is kept the same, and the discharge current and gas flow are reduced to generate a 1.2-kV, 1.43-Å beam. In this mode, the thruster produces 79 mN of thrust. Since the beam voltage remains unchanged for the high-power mode and the thruster mass utilization efficiency is nearly the same, the specific impulse degrades only slightly compared to the high-power mode, to about 3,400 sec.

HALL-EFFECT THRUSTERS

A typical propellant for a Hall thruster is a high-molecular-weight inert gas such as xenon. A power processor is used to generate an electrical discharge between a cathode and an annular anode, through which the majority of propellant is injected. A critical element of the device is the incorporation of a radial magnetic field, which serves to impart an azimuthal drift to the electrons coming from the cathode and to retard their flow to the anode. The azimuthally drifting electrons collide with the neutral xenon, ionizing it. The xenon ions are then accelerated electrostatically from the discharge chamber by the electric potential maintained across the electrodes by the power processor. The velocity of the exiting ions, and hence the Isp, is governed by the voltage applied to the discharge power supply and is typically 15,000-16,000 m/sec at 300 V.

1.5 TWO-DIMENSIONAL TRAJECTORIES OF ROCKETS AND MISSILES

1.5.1 MULTI-STAGE ROCKETS AND TWO STAGE MULTI-STAGE ROCKETS

A multistage (or multi-stage) rocket is, like any rocket, propelled by the recoil pressure of the burning gases it emits as it burns fuel. What characterizes it as "multi-stage" is that it successively jettisons one or more stages as they become empty. It is effectively one or more rockets (stages) stacked on top of or attached next to each other ("parallel staging"); in order to reduce the total amount of mass which needs to be accelerated to the final speed/height. Generally each stage consists of one or more motors, plus fuel and oxidizer tanks for a liquid rocket or the casing for a solid rocket. In rocketry, this concept is known as staging. Solid or liquid rocket Boosters are often used for parallel staging schemes and all motors are ignited at launch.

In other words, a multistage rocket, or step rocket is a rocket that uses two or more stages, each of which contains its own engines and propellant. A tandem or serial stage is mounted on top of another stage; a parallel stage is attached alongside another stage. The result is effectively two or more rockets stacked on top of or attached next to each other. Taken together these are sometimes called a launch vehicle. Two-stage rockets are quite common, but rockets with as many as five separate stages have been successfully launched. By jettisoning stages when they run out of propellant, the mass of the remaining rocket is decreased. This staging allows the thrust of the remaining stages to more easily accelerate the rocket to its final speed and height.

In serial or tandem staging schemes, the first stage is at the bottom and is usually the largest, the second stage and subsequent upper stages are above it, usually decreasing in size. In parallel staging schemes solid or liquid rocket boosters are used to assist with lift-off. These are sometimes referred to as "stage 0". In the typical case, the first-stage and booster engines fire to propel the entire rocket upwards. When the boosters run out of fuel, they are detached from the rest of the rocket (usually with some kind of small explosive charge) and fall away. The first stage then burns to completion and falls off. This leaves a smaller rocket, with the second stage on the bottom, which then fires. Known in rocketry circles as staging, this process is repeated until the final stage's motor burns to completion. In some cases with serial staging, the upper stage ignites before the separation-the interstage ring is designed with this in mind, and the thrust is used to help positively separate the two vehicles. A rocket must travel at a speed of 11.416km/s to escape earth's atmosphere.

The main reason for multi-stage rockets and boosters is that once the fuel is exhausted, the space and structure which contained it and the motors themselves are useless and only add weight to the vehicle which slows down its future acceleration. By dropping the stages which are no longer useful to the mission, the rocket lightens itself. The thrust of subsequent stages is able to provide more acceleration than if the earlier stage were still attached, or a single, large rocket would be capable of. When a stage drops off, the rest of the rocket is still traveling near the speed that the whole assembly reached at burn-out time. This means that it needs less total fuel to reach a given velocity and/or altitude.

A further advantage is that each stage can use a different type of rocket motor each tuned for its particular operating conditions. Thus the lower-stage motors are designed for use at atmospheric pressure, while the upper stages can use motors suited to near vacuum conditions. Lower stages tend to require more structure than upper as they need to bear their own weight plus that of the stages above them, optimizing the structure of each stage decreases the weight of the total vehicle and provides further advantage.

On the downside, staging requires the vehicle to lift motors which are not yet being used, as well as making the entire rocket more complex and harder to build. In addition, each staging event is a significant point of failure during a launch, with the possibility of separation failure, ignition failure, and stage collision. Nevertheless, the savings are so great that every rocket ever used to deliver a payload into orbit has had staging of some sort.

1.5.1.1 TO ACCURATELY PREDICT SPEED AND ALTITUDE FOR YOUR MULTISTAGE ROCKET FROM WEIGHT, DIAMETER, MOTOR THRUST AND IMPULSE.

FIRST STAGE

The equations to find velocity after first stage boost are the same as single stage boost, that is

$$k = \frac{1}{2}\rho C_d A \qquad t = \frac{I}{T}$$

$$q = \sqrt{\frac{T - mg}{k}} \qquad v = q\frac{1 - e^{-xt}}{1 + e^{-xt}}$$

$$x = \frac{2kq}{m} = 2\frac{\sqrt{(T - mg)\cdot k}}{m} \qquad y_1 = \frac{-m}{2k}\ln\left(\frac{T - mg - kv^2}{T - mg}\right)$$

Note: the distance travelled during the first stage boost "y1" instead of "yb". the second stage distance "y2", the third stage distance "y3" and in general, the boost stages are now called "yn".

UPPER STAGES

For each stage after the first, you use a generalized version of these equations. First, set

- v0 equal to the end velocity (v) from the last stage
- m equal to the mass of the rocket after the previous stage has been ejected
- I equal to the impulse for the new stage
- T equal to the thrust for the new stage
- k stays the same as it was for the first stage

The following equations don't change but you recalculate with the new values of m, I and T

$$q = \sqrt{\frac{T - mg}{k}}$$

$$x = \frac{2kq}{m}$$

$$t = \frac{I}{T}$$

Incidentally, the term "q", which is used because it makes the computation easier, also happens to be the terminal velocity for the rocket under thrust. So once you've computed q, you know how close your rocket comes to reaching terminal velocity (the point at which the wind resistance equals thrust, that is, you can't GO faster than this). For single stage rockets this is no big deal because you never come close, but for multistage rockets, you may very nearly reach terminal velocity during boost.

The next term, s, is new for the upper stages:

$$s = \frac{q + v_0}{q - v_0}$$

Then the next two equations are slightly different for the upper stages. Notice that if v0 = 0, these equations become the same as the first stage, or single stage, equations, exactly as they should:

$$v = q\frac{s - e^{-xt}}{s + e^{-xt}}$$

$$y_n = \frac{-m}{2k}\ln\left(\frac{T - mg - kv^2}{T - mg - kv_0^2}\right)$$

The velocity v that you calculate here is the actual velocity of the rocket at the end of the boost phase. The altitude yn is the distance covered only during this boost phase, that is, you have to add up all the y's to get the total altitude the rocket will go.

COASTING

The coasting equations are the same as before with the single stage rocket, that is,

$$y_c = \frac{m}{2k}\ln\left(\frac{mg + kv^2}{mg}\right)$$

$$q_a = \sqrt{\frac{mg}{k}}$$

$$q_b = \sqrt{\frac{gk}{m}}$$

$$t_a = \frac{\tan^{-1}\left(v/q_a\right)}{q_b}$$

Then the total altitude is the sum of each of the altitudes you calculated above, for example, for a three stage rocket the total altitude reached is yt = y1 + y2 + y3 + yc . The coasting time is given by ta, and the total time to apogee is the sum of the burn times plus the coasting time.

1.5.2 VEHICLE SIZING

For initial sizing, the rocket equations can be used to derive the amount of propellant needed for the rocket based on the specific impulse of the engine and the total impulse required in N*s. The equation is

$$m_p = I_{tot}/(g * I_{sp})$$

where g is the gravity constant of the planet (which is Earth in most cases). This also enables the volume of storage required for the fuel to be calculated if the density of the fuel is known, which is almost always the case when designing the rocket stage. The volume is yielded when dividing the mass of the propellant by its density. Asides from the fuel required, the mass of the rocket structure itself must also be determined, which requires taking into account the mass of the required thrusters, electronics, instruments, power equipment, etc.

These are known quantities for typical off the shelf hardware that should be considered in the mid to late stages of the design, but for preliminary and conceptual design, a simpler approach can be taken. Assuming one engine for a rocket stage provides all of the total impulse for that particular segment, a mass fraction can be used to determine the mass of the system. The mass of the stage transfer hardware such as initiators and safe-and-arm devices are very small by comparison and can be considered negligible. For modern day solid rocket motors, it is a safe and reasonable assumption to say that 91 to 94 percent of the total mass is fuel. It is also important to note there is a small percentage of "residual" propellant that will be left stuck and unusable inside the tank, and should also be taken into consideration when determining amount of fuel for the rocket.

A common initial estimate for this residual propellant is five percent. With this ratio and the mass of the propellant calculated, the mass of the empty rocket weight can be determined. Sizing rockets using a liquid bipropellant requires a slightly more involved approach because of the fact that there are two separate tanks that are required: One for the fuel, and one for the oxidizer. The ratio of these two quantities is known as the mixture ratio, and is defined by the equation.

$$O/F = m_{ox} / m_{fuel}$$

Where m_{ox} is the mass of the oxidizer and m_{fule} is the mass of the fuel. This mixture ratio not only governs the size of each tank, but also the specific impulse of the rocket. Determining the ideal mixture ratio is a balance of compromises between various aspects of the rocket being designed, and can vary depending on the type of fuel and oxidizer combination being used. For example, a mixture ratio of a bipropellant could be adjusted such that it may not have the optimal specific impulse, but will result in fuel tanks of equal size.

This would yield simpler and cheaper manufacturing, packing, configuring, and integrating of the fuel systems with the rest of the rocket, and can become a benefit that could outweigh the drawbacks of a less efficient specific impulse rating. But suppose the defining constraint for the launch system is volume, and a low density fuel is required such as hydrogen. This example would be solved by using an oxidizer-rich mixture ratio, reducing efficiency and specific impulse rating, but will meet a smaller tank volume requirement.

1.5.3 TRADE-OFF RATIOS

REMAINING PART OF THE NOTES WILL BE PUBLISHED SOON

UNIT-II

Atmospheric Reentry

INTRODUCTION

2.1 REENTRY DYNAMICS

- Most flights out of earth's atmosphere are designed to be one-way.
- For satellite in LEO orbits re-entry will occur long after useful lifetime of the satellite is exceeded and in case of HEO, re-entry usually never occurs.
- However, there are missions which require surviving re-entry. These missions include:

1. Ballistic missiles (ICBMS, IRBMS)
2. Planetary probes
3. Manned missions

Historically re-entry problem first appeared in German V-2 program where the rockets would explode upon entering atmosphere (the first reentry designs to be tested were ballistic missiles war heads).

WHAT IS RE-ENTRY A PROBLEM?

The problem is the amount of mechanical energy/mass associated with the vehicle.

If we take the orbital velocity 'V' at 8 km/s, the kinetic energy per unit mass of the orbiting object is

$$= \frac{1}{2}\left(8\times10^{3}\right)^{2} = 3.2\times10^{7}\ m^{2}/s^{2}$$

Kinetic Energy/mass~32MJ/kg

Most of this energy has to be removed away from the vehicle for safe return back to earth. If we consider all the heat goes in heating the vehicle mass then

$$C_p\Delta T = 32\times10^{6} \qquad \Delta T = \frac{32\times10^{6}}{C_p}$$

If material is Titanium, then $C_P = 500 J/kgK$ Hence,

$$\Delta T = \frac{32\times10^{6}}{500} = 64{,}000\ K$$

No material can withstand such high temperature

Let us look at the thermal properties of some materials shown in the Table below. It can be seen that most of the materials (except carbon) would vaporize at such temperatures and heating.

Element	Heat of Vaporisation (MJ/kg)	Specific heat, C_P J/kg-K	Melting point (K)
Al	10.2	910	933
Tungsten	4.5	130	3695
Titanium	2.8	540	1941
Silicon	13.6	710	1683
Carbon	59.5	710	3820

Reference: Introduction to flight – Anderson

One way to see it is – The energy with the vehicle is the energy deposited by the launch vehicle. Now to stop or decrease the velocity of space craft will require powerful boosters of size comparable to the launch vehicle stages. Therefore, that option is prohibitive / not possible. Retro rockets are used to decrease the velocity to some extent (which decreases the altitude, h) but not for drastic change.

In addition to concern of high temperature due to aerodynamics heating, the re-entry also poses challenge of enormous structural load on the vehicle during its deceleration through the atmosphere.

2.2 RE-ENTRY DYNAMICS

For quantitative assessment of re-entry deceleration we look at the general equation of motion of a re-entering vehicle. The motion of the re-entry vehicle is described in the body fixed coordinates. Since this frame moves and rotates with the body two other frames of reference are needed to completely describe the motion. The three frames of reference and the vehicle trajectory are show in Fig.2.1. Station '1' and '2' are two locations on the vehicle trajectory.

The three frames of reference shown in fig. 1 are:

(a) Z, X axes attached to the body with 'X' axis directed along the motion of the body. So the velocity vector is in the direction of 'X'.

(b) Local frame moving with the body with 'Z_H' axis pointing in the direction of Earth's gravity and 'X_H' axis along the local horizontal. The axis 'X' makes an angle 'γ' with 'X_H'.

(c) A frame fixed at the center of Earth. Axis 'Z_E' makes an angle 'θ' with Z_H

The forces acting on the re-entry vehicle at any location on the trajectory are shown in fig. 2.2

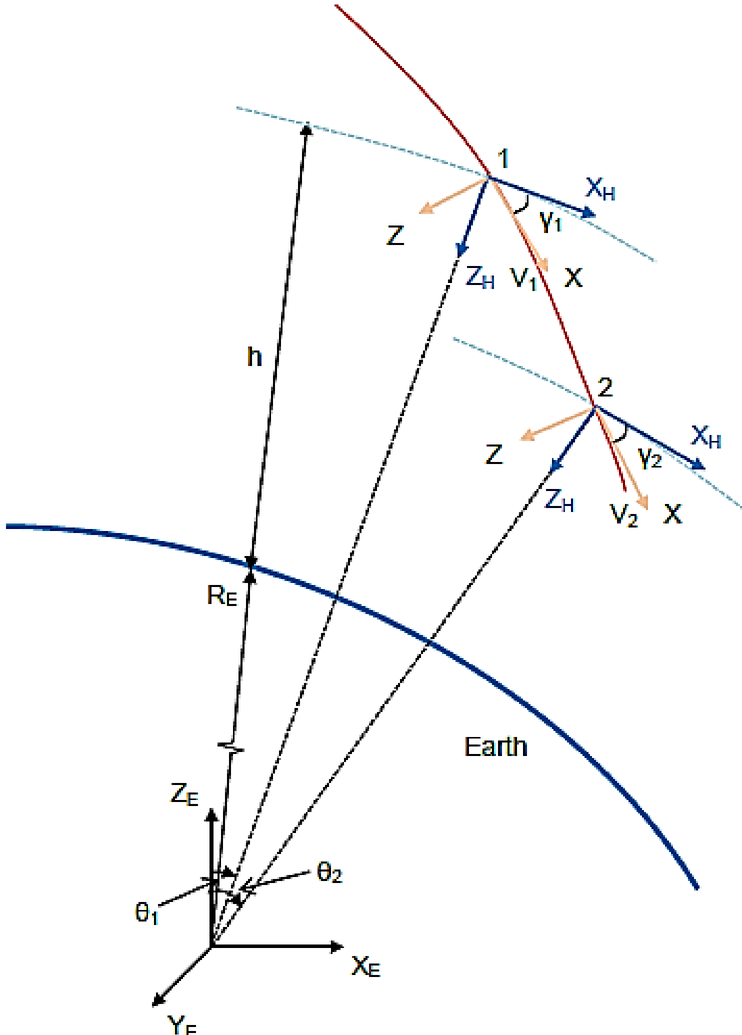


Figure 2.1

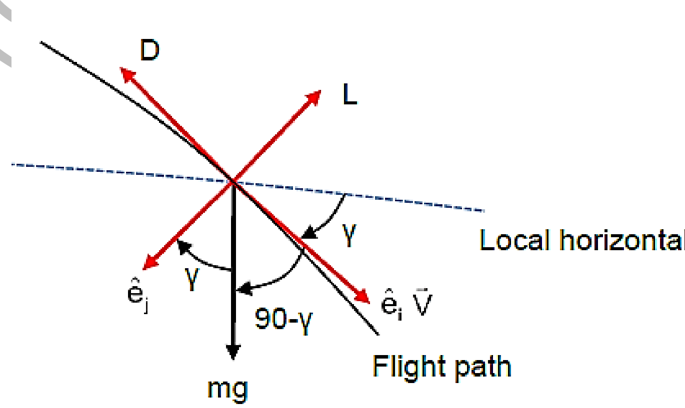


Figure 2.2

Deceleration of the re-entry body can be written as

$$\vec{T}+\vec{A}+\vec{W} = m\frac{d\vec{V}}{dt}$$

Where thrust, $\vec{T}=0$

The aerodynamic force, $\vec{A}=\vec{L}+\vec{D}$ (due to lift and drag)

$$\vec{A} = -D\hat{e}_i - L\hat{e}_j$$

and the weight vector, $\vec{W} = mg\ \sin\gamma\,\hat{e}_i + mg\ \cos\gamma\,\hat{e}_j$

The re-entry vehicle velocity vector can be written as $\vec{V} = V\,\hat{e}_i$

Therefore, $\frac{d\vec{V}}{dt}\Big| = \frac{dV}{dt}\hat{e}_i + V\frac{d\hat{e}_i}{dt} = \frac{dV}{dt}\hat{e}_i + V\left(\dot{\gamma}+\dot{\theta}\right)\hat{e}_j$

Now as $\frac{d\hat{e}_i}{dt} = \dot{\hat{e}}_i = \left(\dot{\gamma}+\dot{\theta}\right)\hat{e}_j$

$$m\left[\frac{dV}{dt}\hat{e}_i + V\left(\dot{\gamma}+\dot{\theta}\right)\hat{e}_j\right] = \left[-D+mg\,sin\,\gamma\right]\hat{e}_i + \left[-L+mg\,cos\,\gamma\right]\hat{e}_j$$

$$-D+mg\ sin\,\gamma = m\frac{dV}{dt}$$

or $\frac{dV}{dt} = g\ sin\,\gamma - \frac{D}{m}$ ----(1)

$-L+mg\,cos\,\gamma = mV\left(\dot{\gamma}+\dot{\theta}\right)$ ----(2)

Now

$\frac{dh}{dt} = -Vsin\,\gamma$ and $R\dot{\theta} = -V\cos\gamma$

$L = \frac{1}{2}\rho V^2 C_L S$ and $D = \frac{1}{2}\rho V^2 C_D S$

Equation (1) now becomes $\frac{dV}{dt} = g\sin\gamma - \frac{\rho V^2}{2}\frac{SC_D}{m}$

or $\frac{dV}{dt} = g\sin\gamma - \frac{\rho V^2}{2\beta}$

where $\beta = \frac{m}{SC_D}$ is the ballistic coefficient,

Similarly equation (2) becomes $V\dot{\gamma} = g\cos\gamma - V\dot{\theta} - \frac{\rho V^2}{2}\left(\frac{SC_L}{m}\right)$

$$V\dot{\gamma} = g\cos\gamma - \frac{V^2\cos\gamma}{R} - \frac{\rho V^2}{2}\left(\frac{SC_D}{m}\times\frac{C_L}{C_D}\right)$$

$$V\dot{\gamma} = \left(g - \frac{V^2}{R}\right)\cos\gamma - \frac{\rho V^2}{2\beta}\left(\frac{C_L}{C_D}\right)$$

The equations of motions are

$$\boxed{\frac{dV}{dt} = g\sin\gamma - \frac{\rho V^2}{2\beta}} \quad \text{----(3)}$$

$$\boxed{V\dot{\gamma} = \left(g - \frac{V^2}{R}\right)\cos\gamma - \frac{\rho V^2}{2\beta}\left(\frac{C_L}{C_D}\right)} \text{----(4)}$$

We can note from equation 4 that '$\dot{\gamma}$' can be altered during the flight only if the vehicle has capability for generating lift.

The above equations of motion are more useful when expressed in terms of local density derivative than in terms of time derivative.

Now, consider

$$\frac{dV}{dt} = \frac{dV}{ds}.\frac{ds}{dt} = V\frac{dV}{ds}$$

Since, $sin\,\gamma = \frac{dh}{ds}$

and $\frac{dV}{ds} = \frac{dV}{dh}.\frac{dh}{ds} = -\sin\gamma\frac{dV}{dh}$

$$\frac{dV}{dt} = -\frac{\sin\gamma}{2}\frac{dV^2}{dh} \qquad \text{----(5)}$$

Further as $\rho = \rho_0\, e^{(-gh/RT)} = \rho_0\, e^{-ah}$ where $a = g\,/\,RT$

$$ln\,\rho = -ah$$

$$\frac{d\rho}{\rho} = -a\,dh$$

$$dh = -\frac{d\rho}{a\rho} \qquad \text{----(6)}$$

Consider equation 3, $\frac{dV}{dt} = g\,sin\,\gamma - \frac{\rho V^2}{2\beta}$

Using equations (5) and (6) in (3)

Consider equation 3, $\frac{dV}{dt} = g\,sin\,\gamma - \frac{\rho V^2}{2\beta}$

Using equations (5) and (6) in (3)

$$-\frac{a\rho\sin\gamma}{2}\frac{dV^2}{d\rho} = g\,sin\,\gamma - \frac{\rho V^2}{2\beta}$$

$$\boxed{\frac{dV^2}{d\rho} + \frac{V^2}{\beta a\sin\gamma} = \frac{2g}{\rho a}} \quad \text{---- (7)}$$

Similarly equation (4): $V\dot{\gamma} = \left(g - \frac{V^2}{R}\right)\cos\gamma - \frac{\rho V^2}{2\beta}\left(\frac{C_L}{C_D}\right)$

$$\dot{\gamma} = \frac{d\gamma}{ds}\frac{ds}{dt} = V\frac{d\gamma}{ds} = -V\frac{d\gamma}{dh}\sin\gamma$$

$$dh = -\frac{d\rho}{a\rho}$$

$$\dot{\gamma} = -Va\rho \frac{d(\cos\gamma)}{d\rho}$$

$$-Va\rho \frac{d(\cos\gamma)}{d\rho} = \left(g - \frac{V^2}{R}\right)\cos\gamma \; - \frac{\rho V^2}{2\beta}\left(\frac{C_L}{C_D}\right)$$

$$\boxed{\frac{d(\cos\gamma)}{d\rho} = \frac{1}{2\beta a}\left(\frac{C_L}{C_D}\right) - \left(\frac{Rg}{V^2} - 1\right)\frac{\cos\gamma}{Ra\rho}} \quad \text{---- (8)}$$

<u>Example</u>: Ballistic re-entry without any lift

As $\left(\frac{C_L}{C_D}\right) = 0 \Rightarrow V\dot{\gamma} = \left(g - \frac{V^2}{R}\right)\cos\gamma$

Further in the initial phase $\frac{V^2}{R} \square g$ and if decrease in velocity is not significant, then

$$\dot{\gamma} = 0$$

$\gamma = constant$ (flight path is a straight line)

$$\frac{dV}{dt} = g\sin\gamma - \frac{\rho V^2}{2\beta}$$

Further if $V >> 1 \ \& \ D \propto V^2$ then $D >> W$ or $\frac{\rho V^2}{2\beta} >> g \sin\gamma$

$$\frac{dV}{dt} = -\frac{\rho V^2}{2\beta} \text{ or } \left|\frac{dV}{dt}\right| = \frac{\rho V^2}{2\beta}$$

Since $\frac{dV}{dt} = -\frac{a\sin\gamma}{2}\left(\frac{dV^2}{d\rho}\right)$

$$\frac{dV^2}{d\rho} = \frac{-V^2}{\beta a \sin\gamma}$$

$$\int_{V_0}^{V} \frac{dV^2}{V^2} = \frac{-1}{\beta a \sin\gamma}\int_0^{\rho} d\rho$$

$$\log\left(\frac{V^2}{V_0^2}\right) = \frac{-\rho}{\beta a \sin\gamma}$$

$$\Rightarrow V = V_0 e^{-\left(\frac{\rho}{2\beta a \sin\gamma}\right)} \text{ or } V = V_0 e^{-\left(\frac{\rho_0 e^{-ah}}{2\beta a \sin\gamma}\right)}$$

The variation of vehicle velocity with altitude is shown in fig. 2.3. The velocity decreases little up to certain altitude before steeply decreasing with altitude. For the vehicle with higher ballistic coefficient the penetration down into the atmosphere is larger.

The velocity decreases with altitude whereas air density increases with altitude. The vehicle deceleration depends on the product of local velocity squared and local density. Therefore a maximum deceleration is expected to occur at some altitude (fig. 2.4)

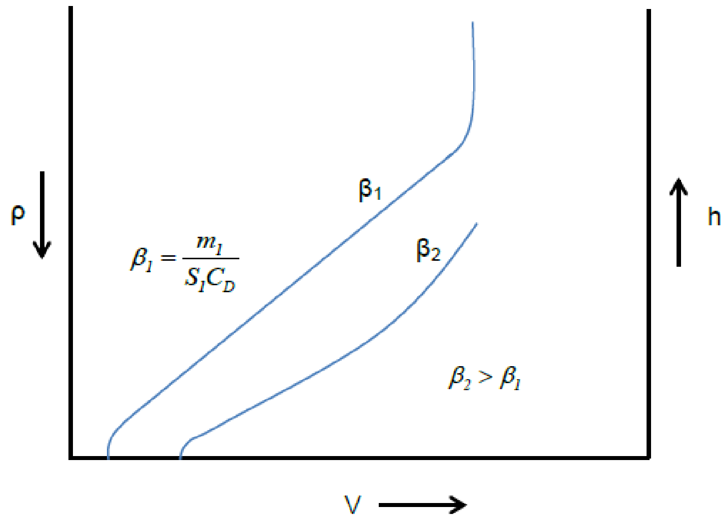


Figure 2.3 Variation of vehicle velocity with altitude

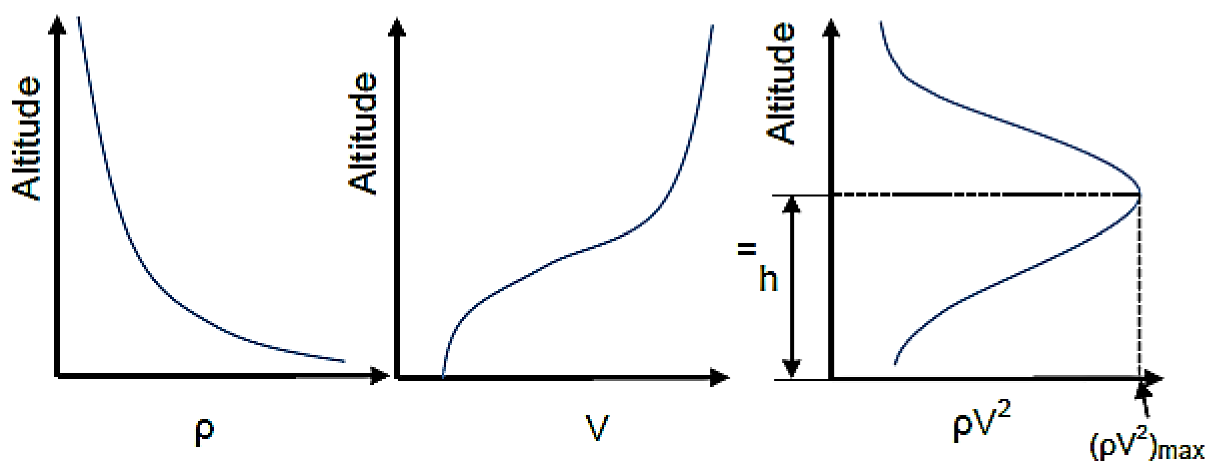


Figure2.4 Existence of altitude at which the deceleration is maximum

Now $\left|\frac{dV}{dt}\right| = \frac{\rho V^2}{2\beta}$

$$\frac{d^2V}{dt^2} = -\frac{1}{\beta}\left[V^2\frac{d\rho}{dt} + 2V\frac{dV}{dt}\rho\right] \text{ as } \frac{dV}{dt} = -\frac{\rho V^2}{2\beta}$$

At the maximum or minimum, $\frac{d^2V}{dt^2} = -\frac{1}{2\beta}\left[-2\rho V \frac{\rho V^2}{2\beta} + V^2 \frac{d\rho}{dt}\right] = 0$

$$\frac{d\rho}{dt} = \frac{\rho^2 V}{\beta}$$

$$\frac{d\rho}{dt} = -a\rho_0 e^{-ah}\frac{dh}{dt} = -a\rho\frac{dh}{dt}$$

& $\frac{dh}{dt} = -V\sin\gamma$ (as $dh = -ds\sin\gamma$)

$$\frac{d\rho}{dt} = a\rho V \sin\gamma$$

$$\frac{\rho V^2}{\beta} = a\rho V \sin\gamma$$

$$\boxed{\rho\Big|_{\left(\frac{dV}{dt}\right)_{max}} = \beta a \sin\gamma}$$

$\rho_0 e^{-ah} = \beta a \sin\gamma$ at $\left(\frac{dV}{dt}\right)_{max}$

or

$$\boxed{h\Big|_{\left(\frac{dV}{dt}\right)_{max}} = \frac{1}{a}\ln\left(\frac{\rho_0}{\beta a \sin\gamma}\right)}$$

$$V\Big|_{\left(\frac{dV}{dt}\right)_{max}} = V_0 e^{-\left(\frac{\beta a \sin\gamma}{2\beta a \sin\gamma}\right)}$$

$$\boxed{V\Big|_{\left(\frac{dV}{dt}\right)_{max}} = V_0 e^{-\left(\frac{1}{2}\right)}}$$

$$\left|\frac{dV}{dt}\right|_{max} = \frac{\rho V^2}{2\beta}\Bigg|_{\left(\frac{dV}{dt}\right)_{max}} = \frac{\beta a \sin\gamma V_0^2 e^{-1}}{2\beta}$$

$$\boxed{\left|\frac{dV}{dt}\right|_{max} = \frac{V_0^2 a \sin\gamma}{2e}}$$

It is interesting to note that $\left|\frac{dV}{dt}\right|_{max} \propto V_0^2, \sin\gamma \,\&\, a$ but independent of β i.e. $\frac{m}{SC_D}$

2.3 RE-ENTRY TYPES

Based on re-entry strategy there can be three categories of re-entry:

2.3.1 BALLISTIC REENTRY

Here the vehicle has little or no aerodynamic lift. It falls through the atmosphere under the influence of drag and gravity, impacting the surface at point in Fig. 2.5. The impact point is predetermined by the conditions at first entry to the atmosphere. The pilot has no control over his or her landing position during this ballistic trajectory. It literally is the same as falling to the surface. Before the space shuttle, virtually all entries of existing space vehicles were ballistic

2.3.2 SKIP REENTRY

This strategy uses atmosphere to slow down. There are two types here depending on whether the vehicle has lifting surface or not.

(I) with lifting surfaces (L/D between 1 & 4): the vehicle grazes the atmosphere then performs pitch-up to go out of atmosphere (slowing down a bit). This process is repeated several times until vehicle finally slows down enough to penetrate the atmosphere (originally proposed by Eugene Sanger, a Swiss Engineer as a concept for intercontinental bomber during WWII)

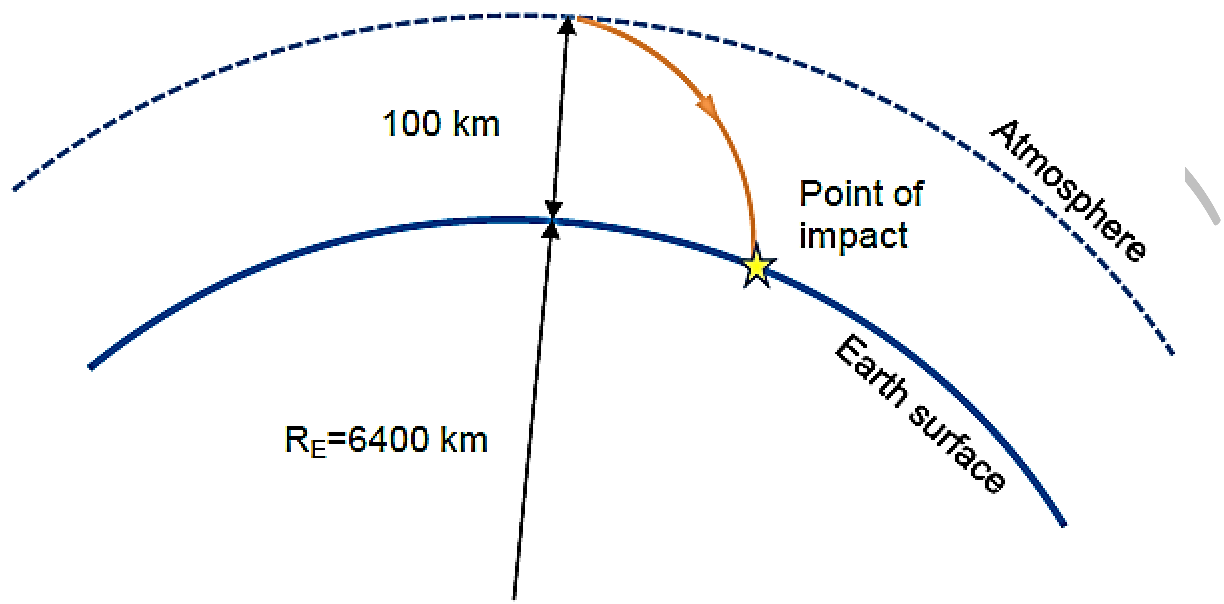


Figure 2.5 Ballistic reentry from space

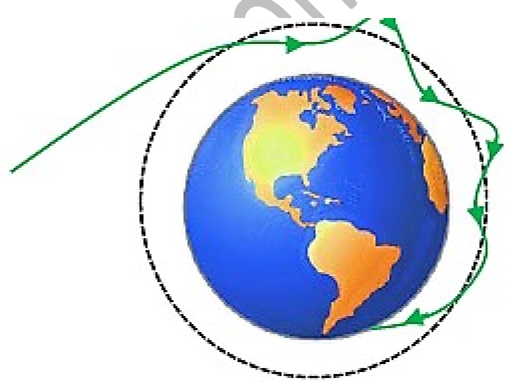


Figure 2.6 Skip Reentry for the spacecraft with lifting surfaces

(II) Without lifting surface (proposed by Walter Hohmann): Hohmann braking ellipses (aero braking or aero capture, see Fig. 2.7). A returning probe/spaceship will approach earth with orbital energy $\varepsilon T>0$ (i.e. hyperbolic path). The vehicle when passes (skims) through the atmosphere is said to be captured if the orbital energy $\varepsilon T<0$. Due to this fact the first passage is critical and it produces the highest heating. Once captured subsequent the trajectory of the vehicle about earth will be ellipses with major axis reducing with every pass through the atmosphere. Small maneuvers at apogee can be carried out for controlling perigee location

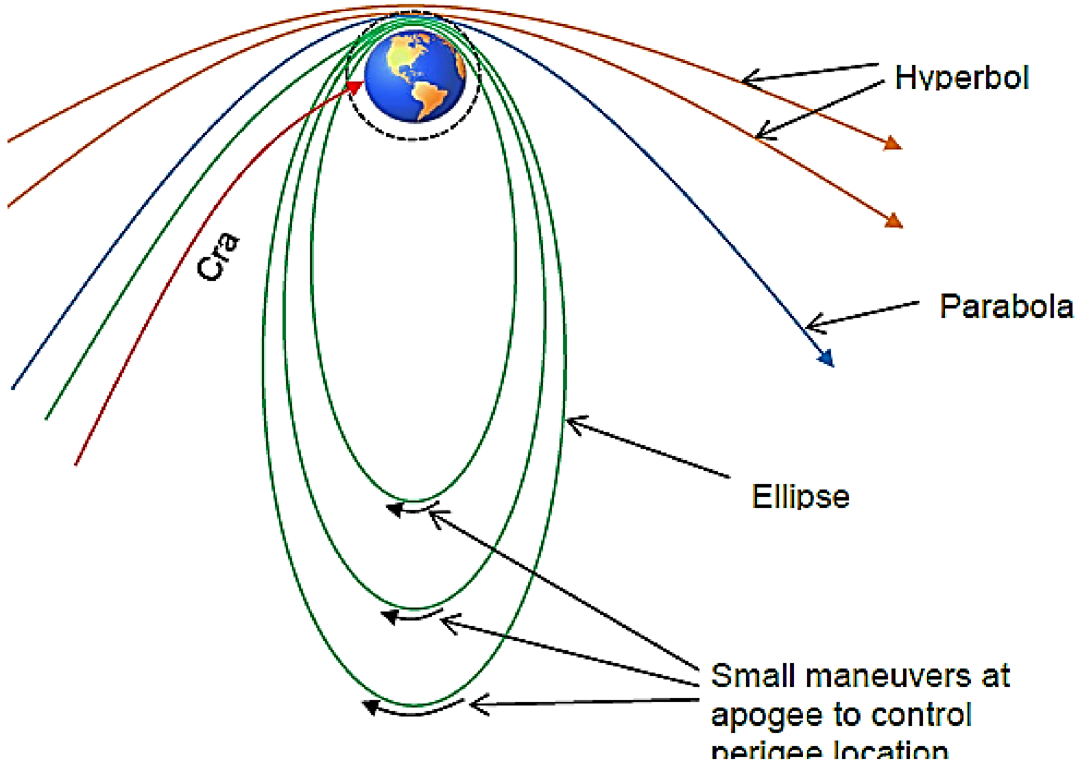


Figure 2.7 Aero capture re-entry

2.3.3 Glide reentry (Lifting body L/D – 4 or greater): For vehicles with lifting surfaces capable of producing high lift one can have controlled deceleration as the vehicle descends. Here the re-entry begins with very shallow flight path angle and high angle of attack.

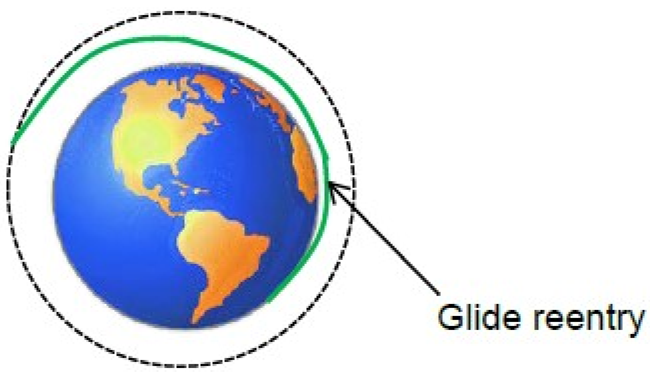


Figure 2. 8 Glide reentry

2.3 RE-ENTRY CORRIDOR

From the above discussion one can understand that if deceleration is too high then heating and structural load are immensely high and this will lead to failed re-entry by destruction of the vehicle. On the other hand, if the deceleration is too small, then in case or earth returning vehicle, the vehicle may be lost to space forever. Therefore, adequate deceleration is required to bring the vehicle back to earth safely. The region determined by bounds of trajectory which will lead to safe deceleration of vehicle is referred to as re-entry corridor.

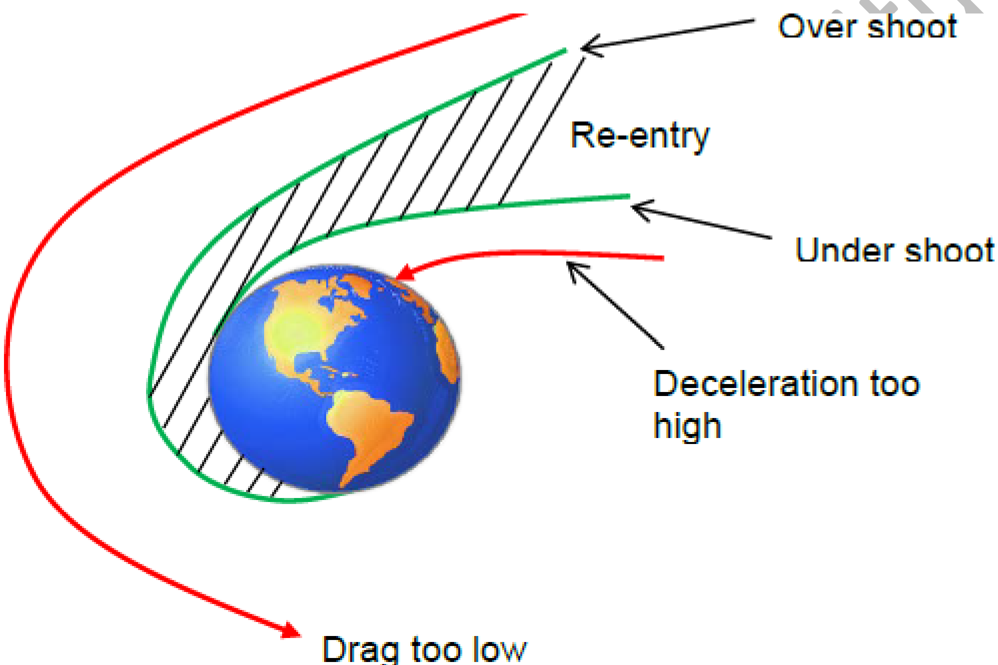


Figure 2.9 Re-entry corridor

2.4 EXAMPLE OF BALLISTIC RE-ENTRY:

Let us consider ballistic reentry for an object with the following specifications:

$V_E = 8\,km/s$

$\gamma = 10^\circ$

$\beta = \dfrac{m}{SC_D} = 5000 kg/m^2$

$T = 288\,K,\ a = \dfrac{g_0}{RT} = \dfrac{9.81}{287 \times 288} = 0.000118\,m^{-1}$

We need to find 'h' at which deceleration is maximum, what is this maximum deceleration, velocity at this altitude and velocity of the vehicle when it reaches surface of earth.

1) $\rho\Big|_{\left(\frac{dV}{dt}\right)_{max}} = \beta a \sin\gamma = 5000 \times 0.000118 \sin 10^\circ$

$= 0.1025 kg/m^3$

$h\Big|_{\left(\frac{dV}{dt}\right)_{max}} = \dfrac{1}{a} ln\left(\dfrac{\rho_0}{\rho}\right)_{\left(\frac{dV}{dt}\right)_{max}} = \dfrac{1}{0.000118} ln\left(\dfrac{1.225}{0.1025}\right) = 21028\,m$

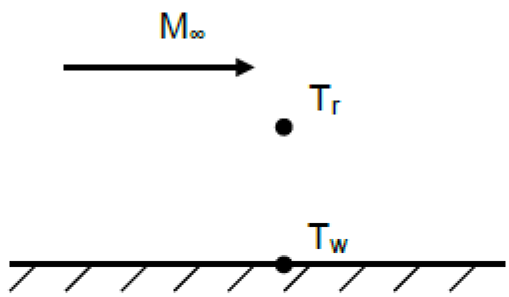
2) $\left|\dfrac{dV}{dt}\right|_{max} = \dfrac{a \sin\gamma V_0^2}{2e} = \dfrac{8000^2 \times 0.000118 \times \sin 10^\circ}{2 \times 2.718} = 241.24\,m/s^2 = 24.6g$

3) $V\Big|_{\left(\frac{dV}{dt}\right)_{max}} = V_0 e^{-\left(\frac{1}{2}\right)} = \dfrac{8000}{\sqrt{2.718}} = 4.852\,km/s$

4) $\dfrac{V}{V_0}\Big|_{R=R_E} = e^{-\left(\frac{\rho_0}{2\beta a \sin\gamma}\right)} = e^{-\left(\frac{1.225}{2\times5000\times0.000118\times\sin 10^\circ}\right)} = 0.00253$

$V = 20.26\,m/s$

2.5 REENTRY HEATING



$T_r \square T_0$ where T_r is the recovery temperature close to the surface.

The local heat flux to the surface of vehicle can be written as $\dot{q} = h(T_r - T_w)$

$$(T_r - T_0) \square (T_0 - T_w) = (T_\infty - T_w) + T_\infty \frac{(\gamma - 1)}{2} M_\infty^2$$

$$(T_\infty - T_w) \square T_\infty \frac{(\gamma - 1)}{2} M_\infty^2$$

$$M_\infty^2 = \frac{V^2}{a^2} = \left(\frac{V}{\sqrt{\gamma R T}} \right)^2 = \frac{V^2}{\gamma R T_\infty}$$

$$C_p = \frac{\gamma R}{\gamma - 1} \, or \, \gamma R = C_p(\gamma - 1)$$

$$M_\infty^2 = \frac{V^2}{T_\infty C_p(\gamma - 1)}$$

$$(T_\infty - T_w) \cong T_\infty \frac{(\gamma - 1)}{2} \times \frac{V^2}{T_\infty C_p(\gamma - 1)} = \frac{V^2}{2C_p}$$

$$\dot{q} = \frac{hV^2}{2C_p}$$

Note: The box symbol in the equation are by mistake it should be ≈

Recall Reynold's analogy which relates momentum transport and heat transport for fluid with Pr~1

$$S_t = \frac{Nu}{Re.Pr} = \frac{h}{\rho V C_p} \square \frac{C_f}{2} \text{ (for } Pr = 1\text{)}$$

or the heat transfer coefficient may then be obtained as $h = \frac{\rho V C_p C_f}{2}$

$$\dot{q} = \frac{\rho V C_p C_f}{2} \frac{V^2}{2C_p} = \frac{1}{4} C_f \rho V^3$$

$$\boxed{\dot{q} = \frac{1}{4} C_f \rho V^3}$$

$$S\dot{q} = \frac{dQ}{dt} = \frac{dQ}{dV}\frac{dV}{dt}$$

$$\frac{dV}{dt} = -\frac{\rho V^2}{2\beta}$$

$$-\frac{1}{4} C_f \rho V^3 S = \frac{dQ}{dV} \frac{\rho V^2 S C_D}{2m}$$

$$\frac{dQ}{dV} = -\frac{1}{4}\left(\frac{C_f}{C_D}\right) Vm$$

$$dQ = -\frac{1}{2}\left(\frac{C_f}{C_D}\right) d\left(\frac{1}{2} m V^2\right)$$

$$Q = -\frac{1}{2}\left(\frac{C_f}{C_D}\right)\frac{1}{2} m V_E^{\,2}$$

$$Q \propto \frac{1}{2} m V_E^{\,2}$$

$$Q \propto \left(\frac{C_f}{C_D}\right)$$

Where $C_D = C_{Dp} + C_f$

....

CASE 1: SHARP AND SLENDER BODY

$C_{Dp} \square 0$ and $\left(\frac{C_f}{C_D}\right) \square 1$

$$Q_{total} = \frac{1}{4} m V_E^{\,2}$$

CASE 2: SHARP AND SLENDER BODY

$$C_D \square C_{Dp}$$

$$Q_{total} = \frac{C_f}{C_{Dp}}\left(\frac{1}{4} m V_E^{\,2}\right)$$

$\frac{C_f}{C_{Dp}} << 1$ this implies $Q_{total,blunt} << Q_{total,slender}$

Thus for reducing re-entry heating blunt nosed vehicles are preferred.

Note: The box symbol in the equation are by mistake it should be ≈

UNIT-III

Fundamentals of Orbit Mechanics, Orbit Maneuvers

ASTRODYNAMICS / ORBITAL MECHANICS

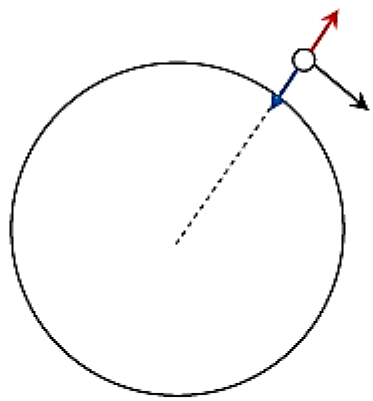
Study of the motion of man-made objects in space subject to both natural and artificially induced forces (different from the parent science celestial mechanics, where there is no human intervention).

An astrodynamicist's job is to determine the trajectory to accomplish the desired mission goals within the constraints of limits imposed by physics and launch vehicle performance.

3.1 A REVIEW OF WHAT WE KNOW FROM HIGH SCHOOL PHYSICS

The orbital velocity:

Gravitational force = $GmM/R^2 = mg$ Centrifugal force = mV^2/R



$mg = mV^2/R$

$V_{orbital} = \sqrt{(gR)}$

$= 8\ km/s$

Escape velocity:

$V_{orbital} = \sqrt{(2gR)} = 11.2 km/s$

Equation of Motion:

The equation of motion for single body about a central gravitational field is given by

$$\vec{F} = m\ddot{\vec{r}} = -\frac{GMm}{|\vec{r}|^3}\vec{r}$$

or $$\ddot{\vec{r}} + \frac{GM}{|\vec{r}|^3}\vec{r} = \ddot{\vec{r}} + \frac{\mu}{|\vec{r}|^3}\vec{r} = 0 \text{ where } \mu = GM$$

<u>Kepler's law of planetary motion (~1600AD, Germany based on data of Tacho Brahe)</u>

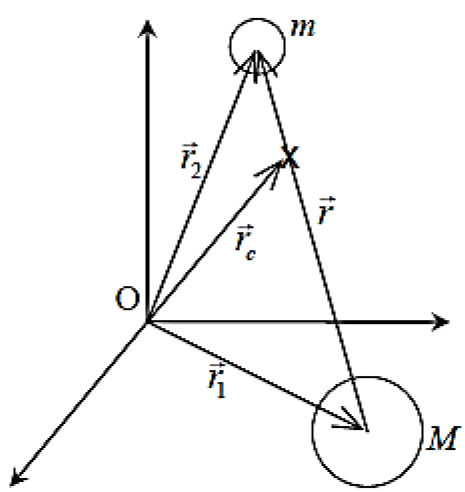
1. Planets move around Sun in elliptical orbits with Sun as one of the foci.
2. The line joining the planet and the Sun sweeps equal area in equal times.
3. The square of the time period of revolution of a planet about the Sun is proportional to the major axis of the ellipse

3.2 FUNDAMENTALS OF ORBITAL MECHANICS

<u>Two-Body motion</u>

The basis of astrodynamics is Newton's universal law of gravitation which describes the force between two point masses M and m separated by a distance $|\vec{r}|$ along the vector $\vec{r}$.

$$F = -\frac{GMm}{|\vec{r}|^3}\vec{r}$$



The equations of motion for the system can be written as:

$$\vec{F}_M = M\ddot{\vec{r}}_1 = -\frac{GMm}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \qquad \text{------(1)}$$

$$\vec{F}_m = m\ddot{\vec{r}}_2 = -\frac{GMm}{\left|\vec{r}_1 - \vec{r}_2\right|^3}\left(\vec{r}_2 - \vec{r}_1\right) \qquad \text{------(2)}$$

The information available forbids determination of all the constants of integration (12 in number) that are obtained upon integration of equations '1' and '2'. However, the relative motion between the two bodies is amenable to solution.

Adding Equn. (1) and Equn. (2)

$$M\ddot{\vec{r}}_1 + m\ddot{\vec{r}}_2 = 0 \qquad \text{-----(3a)}$$

Let the position vector to the center of mass of the *m* and *M* mass system be $\vec{r}_c$

$$\vec{r}_c = \frac{M\vec{r}_1 + m\vec{r}_2}{M + m}$$

Equn. (3) becomes $\boxed{\ddot{\vec{r}}_c = 0}$ -----(3b)

i.e. the centre of mass of the two body system is un-accelerated and thus can serve as the origin of an inertial frame.

Now subtracting $m \times$ Equn. (1) form $M \times$ Equn.(2)

$$Mm\left(\ddot{\vec{r}}_2 - \ddot{\vec{r}}_1\right) = -\frac{GMm(M + m)}{\left|\vec{r}_1 - \vec{r}_2\right|^3}\left(\vec{r}_2 - \vec{r}_1\right)$$

$$\Rightarrow \ddot{\vec{r}} = -\frac{G(M + m)}{\left|\vec{r}\right|^3}\vec{r} = -\frac{\mu}{\left|\vec{r}\right|^3}\vec{r}, \quad where\ \mu = G(M + m) \text{ is called the reduced mass}$$

$$\ddot{\vec{r}} + \frac{\mu}{\left|\vec{r}\right|^3}\vec{r} = 0 \qquad \text{------(4)}$$

This equation is same as the equation of motion for a single mass about a central gravitational field, except that the present equation, equn. (4) describes the motion of body 'm' relative to body M. Further for M>>m the center of mass of the system shifts almost to the center of mass 'M' and thus reduce the two boy problem to that of motion of single body about central gravitational field.

Linear momentum

Linear momentum of mass 'm' and 'M' are $\vec{p}_M = M\dot{\vec{r}}_1$ and $\vec{p}_m = m\dot{\vec{r}}_2$ respectively

Linear momentum of the system $\vec{p} = \vec{p}_M + \vec{p}_m = M\dot{\vec{r}}_1 + m\dot{\vec{r}}_2 = const.$ as $\ddot{\vec{r}}_c = 0$

Angular momentum

The angular momentum of the system is $\vec{H} = \vec{H}_M + \vec{H}_m$ where $\vec{H}_M = \vec{r}_1 \times M\dot{\vec{r}}_1$ and $\vec{H}_m = \vec{r}_2 \times m\dot{\vec{r}}_2$

Now total torque on the system will result in the rate of change of the momentum of the system.

i.e. $\tau = \sum_i \tau_i = \sum_i \vec{r}_i \times F_i = \sum_i \frac{d}{dt}\left(\vec{r}_i \times m_i \dot{\vec{r}}_i\right) = \sum_i \dot{\vec{H}}_i = \dot{\vec{H}}$

$$\vec{\tau} = \sum_i \vec{\tau}_i = \sum_i \vec{r}_i \times \vec{F}_i = \vec{r}_1 \times \frac{GMm}{|\vec{r}|^3}(\vec{r}_1 - \vec{r}_2) + \vec{r}_2 \times \frac{GMm}{|\vec{r}|^3}(\vec{r}_2 - \vec{r}_1) = -\vec{r}_1 \times \vec{r}_2 - \vec{r}_2 \times \vec{r}_1 = 0$$

One can also show that $\dot{\vec{H}} = \sum_i \dot{\vec{H}}_i = \sum_i \frac{d}{dt}\left(\vec{r}_i \times m_i \dot{\vec{r}}_i\right)$

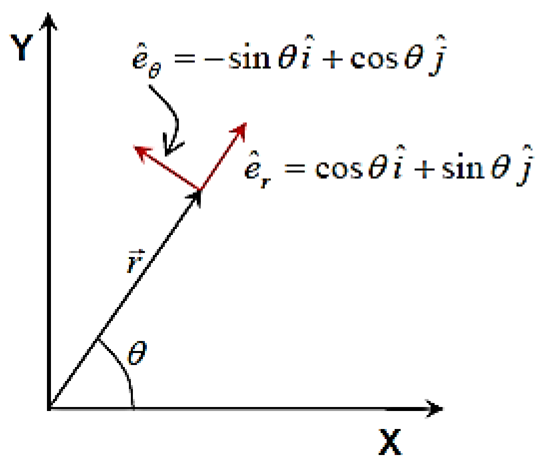
$$\dot{\vec{H}}_M = \dot{\vec{r}}_1 \times M\dot{\vec{r}}_1 + \vec{r}_1 \times M\ddot{\vec{r}}_1 = 0 + \vec{r}_1 \times \frac{GMm}{|\vec{r}|^3}\vec{r} \;\; \& \;\; \dot{\vec{H}}_m = \dot{\vec{r}}_2 \times m\dot{\vec{r}}_2 + \vec{r}_2 \times m\ddot{\vec{r}}_2 = 0 - \vec{r}_2 \times \frac{GMm}{|\vec{r}|^3}\vec{r}$$

$$\dot{\vec{H}} = (\vec{r}_1 - \vec{r}_2) \times M\frac{\mu}{|\vec{r}|^3}\vec{r} = \vec{r} \times \frac{GMm}{|\vec{r}|^3}\vec{r} = 0$$

$\vec{H} = \text{constant vector}$ and perpendicular to the plane containing $\vec{r}$ and $\dot{\vec{r}}$, therefore, the motion described by the bodies will be two-dimensional along the plane orthogonal to the angular velocity vector.

The equation of motion of mass 'm' about 'M' described by the ODE

$\ddot{\vec{r}} + \frac{\mu}{|\vec{r}|^3}\vec{r} = 0,$ *where* $\mu = G(M+m)$ *and* $\vec{r} = \vec{r}_2 - \vec{r}_1$ can be expressed in polar co-ordinates as show below



$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta$$
$$\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$$

$$\vec{r} = r\,\hat{e}_r$$
$$\dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$
$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta$$

Thus equn. (4), i.e. equation of motion in polar co-ordinates becomes

$$\left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta + \frac{\mu}{r^2}\hat{e}_r = 0$$

$$\left(\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2}\right)\hat{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_\theta = 0$$

$$radial\ component : \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0 \qquad and \qquad circumferential\ \ component : 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

Let us look at the circumferential component

The circumferential component can be written as: $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

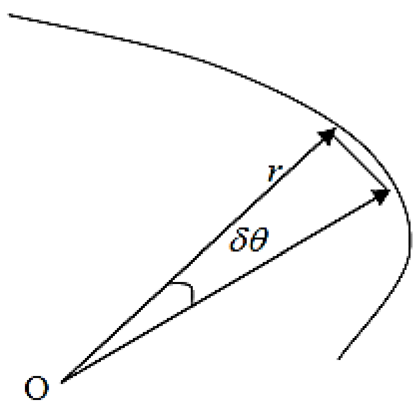
or $r^2\dot{\theta} = const. = h$

where h is angular momentum/mass, or $h = \vec{r} \times \dot{\vec{r}} = \left(r\hat{e}_r\right) \times \left(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta\right) = r^2\dot{\theta}\,\hat{e}_z$

$$\frac{\delta A}{\delta t} = \frac{1}{2} r \times r \frac{\delta\theta}{\delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = const.$$

This is Kepler's second law



Now we look at the radial component

$$radial\ component : \ \ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$$

$$\dot{r}\left(\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2}\right) = 0 \qquad OR \qquad \dot{r}\ddot{r} - \dot{r}\frac{h^2}{r^3} + \dot{r}\frac{\mu}{r^2} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} - \frac{\mu}{r}\right) = 0 \quad OR \quad \frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} - \frac{\mu}{r} = const.$$

$$OR \quad \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{\mu}{r} = const.$$

$$\underbrace{\left(\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right)}_{kinetic\ energy/mass} + \underbrace{\left(-\frac{\mu}{r}\right)}_{potential\ energy/mass} = const. = \underbrace{E_T}_{total\ energy/mass} \quad (Orbital\ Energy)$$

Total energy of the two body system remains a constant.

3.3 EQUATION OF TRAJECTORY

From radial component we get $\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) \quad \ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

$$-\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{h^2}{r^3} + \frac{\mu}{r^2} = 0$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} - \frac{\mu}{h^2} = 0 \quad let\ \frac{1}{r} - \frac{\mu}{h^2} = Z \quad \Rightarrow \frac{d^2Z}{d\theta^2} + Z = 0$$

$$Z = A\cos\theta + B\sin\theta = C\cos(\theta - \psi), \quad where\ C = \sqrt{A^2 + B^2}\ and\ \psi = \tan^{-1}\left(\frac{B}{A}\right)$$

$$\frac{1}{r} = \frac{\mu}{h^2} + C\cos(\theta - \psi) = \frac{\mu}{h^2}(1 + e\cos(\theta - \psi)) \quad where\ e = \frac{Ch^2}{\mu}$$

The equation of trajectory of mass 'm' about 'M' is given by: $r = \frac{h^2/\mu}{(1 + e\cos(\theta - \psi))}$

The equation above represents a class of curves referred to as conic curves. We take a look at the mathematical description of the conic curves next.

3.3.1 THE CONIC SECTION

The family curves obtained when a plane cuts a conical surface is called conics. The different conics are shown below.

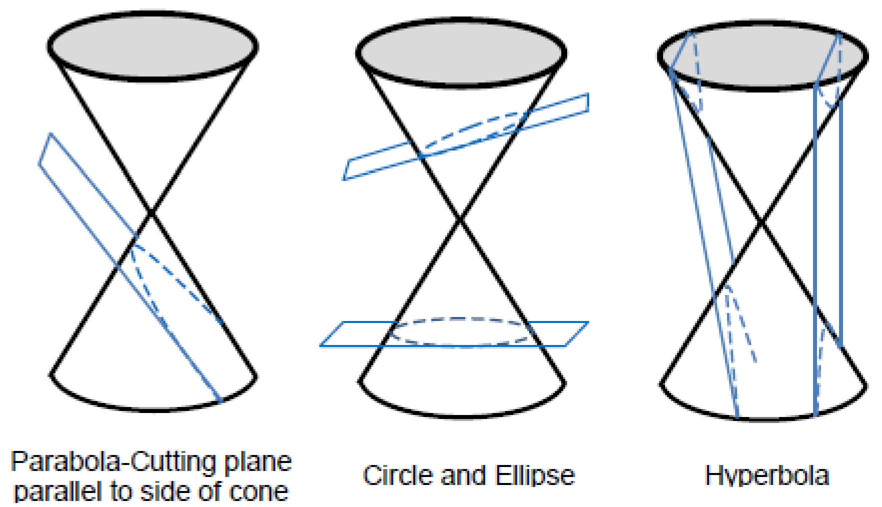


Figure 3. 1

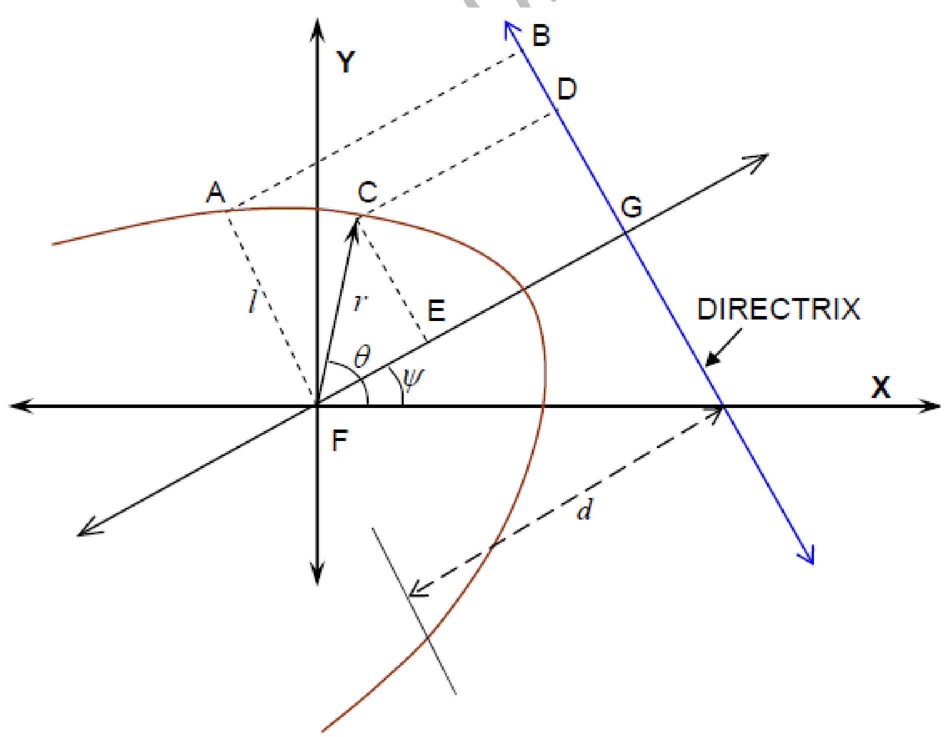


Figure 3. 2

Mathematically defining, a conic is a locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed line is a positive constant. The ratio is the eccentricity of the conic, the fixed point the focus and the fixed line the directrix (Fig. 3.2).

$$r = CF = e\,CD = e\left(FG - FE\right) = e\left(AB - FE\right) = l - er\cos\left(\theta - \psi\right)$$

$$r = \frac{l}{1 + e\cos(\theta - \psi)} = \frac{ed}{1 + e\cos(\theta - \psi)}$$

In the Cartesian coordinate system, the graph of a quadratic equation in two variables is always a conic section, and all conic sections arise in this way. If the equation is of the form

$$ax^2 + 2cxy + by^2 + 2dx + 2ey + c = 0$$

then:

if $c^2 = ab$, the equation represents a parabola

if *cb* and/or *c*0 , the equation represents an ellipse $^2 < ab$ and $a \neq \neq$

if $c^2 > ab$, the equation represents a hyperbola

if $c^2 < ab$ and $a = b$ and $c = 0$, the equation represents a circle

3.3.2 TRAJECTORY CLASSIFICATION

Equation of trajectory for a two body problem $r = \dfrac{h^2/\mu}{\left(1 + e\cos(\theta - \psi)\right)}$

Let $\psi = 0$, then $r = \dfrac{h^2/\mu}{1 + e\cos\theta}$ (see figure below)

$$\underbrace{\left(\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right)}_{kinetic\ energy/mass} + \underbrace{\left(-\frac{\mu}{r}\right)}_{potential\ energy/mass} = const. = \underbrace{E_T}_{total\ energy/mass} \quad (Orbital\ Energy)$$

or $E_T = \dfrac{1}{2}V^2 - \dfrac{\mu}{r} = const.$, where $V^2 = \dfrac{1}{2}\dot{r}^2 + \dfrac{1}{2}r^2\dot{\theta}^2$

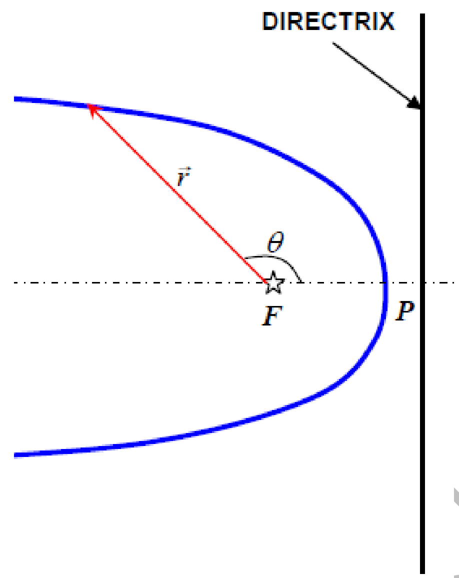


Figure 3.3

Now the constant value of E_T can be evaluated at $\theta = 0$, where $r = r_{\min}$ and $\dot{r} = 0$

$$E_T = \frac{1}{2} r_{\min}{}^2 \dot{\theta}^2 - \frac{\mu}{r_{\min}}, \text{ also } h = r^2\dot{\theta} = const. = r_{\min}{}^2\dot{\theta} \text{ and } r_{\min} = \frac{h^2/\mu}{1+e}$$

$$\therefore \quad E_T = \frac{1}{2} r_{min}^{\ 2} \dot{\theta}^2 - \frac{\mu}{r_{min}} = \frac{1}{2} \frac{h^2}{r_{min}^2} - \frac{\mu}{r_{min}} = \frac{\mu(e-1)}{2r_{min}} = -\frac{\mu^2(1-e^2)}{2h^2}$$

$$E_T = -\frac{\mu^2(1-e^2)}{2h^2} = \frac{1}{2} r^2 \dot{\theta}^2 - \frac{\mu}{r}$$

When,

$e<1 \quad then \quad E_T<0 \qquad -\frac{\mu}{r}\left(\frac{Potential\ Energy}{mass}\right) > \frac{1}{2}V^2\left(\frac{Kinetic\ Energy}{mass}\right)$, the trajectory is an ELLIPSE (Kepler's first law)

$e=1 \quad then \quad E_T=0 \qquad -\frac{\mu}{r}\left(\frac{Potential\ Energy}{mass}\right) = \frac{1}{2}V^2\left(\frac{Kinetic\ Energy}{mass}\right)$, the trajectory is a PARABOLA

$e>1 \quad then \quad E_T>0 \qquad -\frac{\mu}{r}\left(\frac{Potential\ Energy}{mass}\right) < \frac{1}{2}V^2\left(\frac{Kinetic\ Energy}{mass}\right)$, the trajectory is a HYPERBOLA

Case I: Ellipse, $E_T < 0 \quad or \quad e<1 \ \ (for\ e=0,\ circle)$

$$FP = r_{min} = r_P \ and \ FA = r_{max} = r_A$$

$$r_A + r_P = 2a$$

$$\frac{FA}{AD} = e = \frac{FP}{PD} \ therefore,\ FA - FP = r_A - r_P = ae$$

$$r_P = a(1-e) \ and \ r_A = a(1+e)$$

Now trajectory equation: $r = \frac{h^2/\mu}{1+e\cos\theta}$, where $e = \frac{\sqrt{a^2-b^2}}{a}$ and $\frac{h^2}{\mu} = \frac{b^2}{a}$

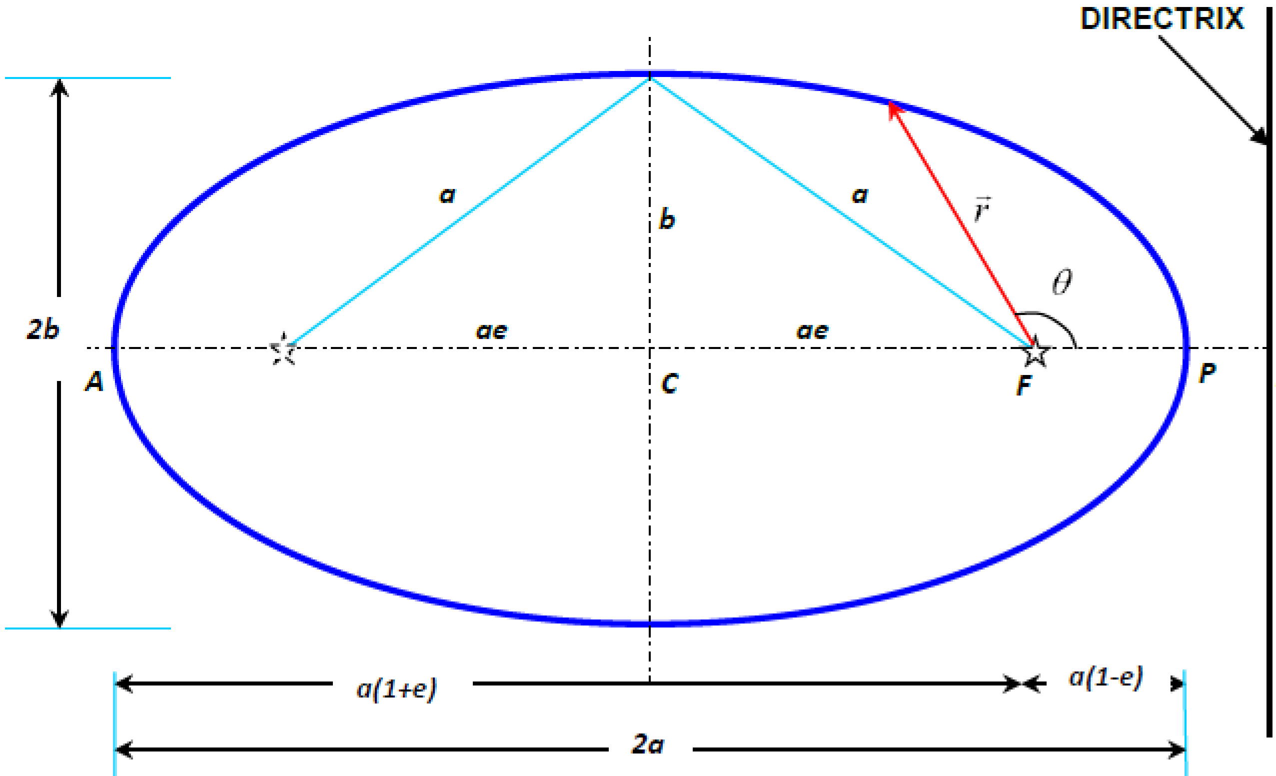


Figure 1

$$r_{\min} = r_P = \frac{h^2/\mu}{1+e} = a(1-e) \text{ and } r_{\max} = r_A = \frac{h^2/\mu}{1-e} = a(1+e) \text{ and } \frac{h^2}{\mu} = a(1-e^2) = \frac{b^2}{a}$$

Orbital energy: $E_T = \frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu^2(1-e^2)}{2h^2} = -\frac{\mu}{2a}$

Velocity at radius r: $V = \sqrt{\frac{\mu}{a}\left(\frac{2a}{r} - 1\right)}$

Therefore velocity at periapsis, $V_P = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)} = \sqrt{\frac{2\mu r_A}{r_P(r_P + r_A)}}$

and velocity at apoapsis, $V_A = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)} = \sqrt{\frac{2\mu r_P}{r_A\left(r_P + r_A\right)}}$

Velocity needed to escape orbit $E_T = 0 = \frac{1}{2}V^2 - \frac{\mu}{r}$

or $V = \sqrt{\frac{2\mu}{r}}$

ΔV_{escape}, excess velocity required to escape orbit from a particular point on the orbit

$$\Delta V_{escape} = \sqrt{\frac{2\mu}{r}} - \sqrt{\frac{\mu}{a}\left(\frac{2a}{r} - 1\right)}$$

Therefore $\Delta V_{escape,P} = \sqrt{\frac{2\mu}{a(1-e)}} - \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)} = \sqrt{\frac{2\mu}{r_P}} - \sqrt{\frac{2\mu r_A}{r_P\left(r_P + r_A\right)}}$

and $\Delta V_{escape,A} = \sqrt{\frac{2\mu}{a(1+e)}} - \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)} = \sqrt{\frac{2\mu}{r_A}} - \sqrt{\frac{2\mu r_P}{r_A\left(r_P + r_A\right)}}$

Also note that $T = \frac{A}{dA/dt} = \frac{\pi ab}{h/2} = \frac{\pi a^2\sqrt{\left(1-e^2\right)}}{\sqrt{a\left(1-e^2\right)\mu}/2} = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$ or $T^2 \sim a^3$ This is Kepler's third law.

3.3.3 TIME OF FLIGHT FOR ELLIPTICAL TRAJECTORIES

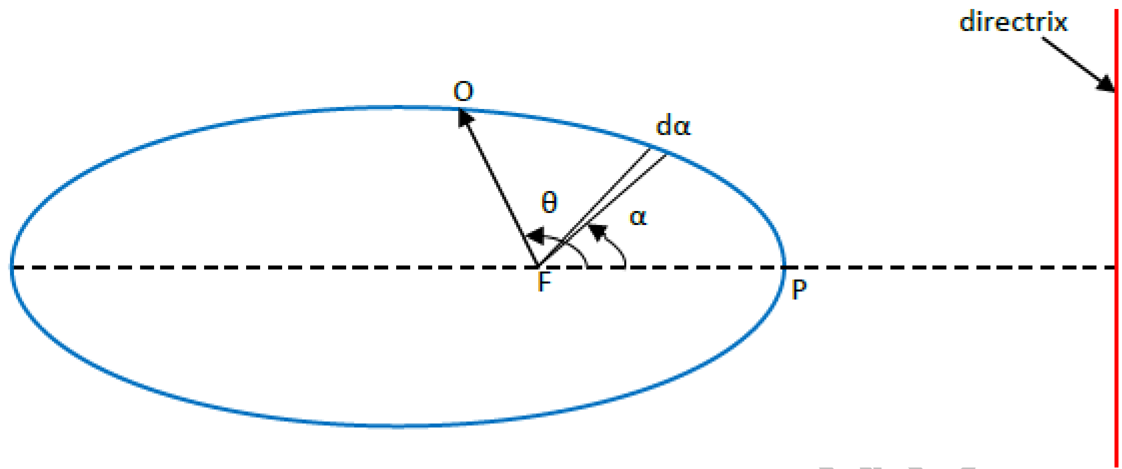


Figure 1

Here, we are interested in the time taken to reach point 'O' starting from the periapsis P.

$$Area\ OFP = A_e = \frac{1}{2}\int r^2\, d\alpha$$

Now

$$\frac{dA}{dt} = \frac{h}{2} = const$$

On rearranging, we get

$$A_e = \int_0^{A_e} dA = \frac{h}{2}\int_{t_0}^{t} dt$$

$$\int_0^{\theta} r^2\, d\alpha = \frac{h}{2}(t - t_0)$$

$$r = r(\alpha) = \frac{a\,(1 - e^2)}{1 + e\cos\alpha}$$

A_e is not so straightforward to evaluate form the above relation. Kepler introduced an easier way to evaluate the same, since an ellipse can be viewed as a projection of a circle (called an Auxiliary Circle, see below) of radius 'a' such that all the distances parallel to the major axis remain the same and the distances perpendicular to the major axis are reduced by a factor 'b/a'.

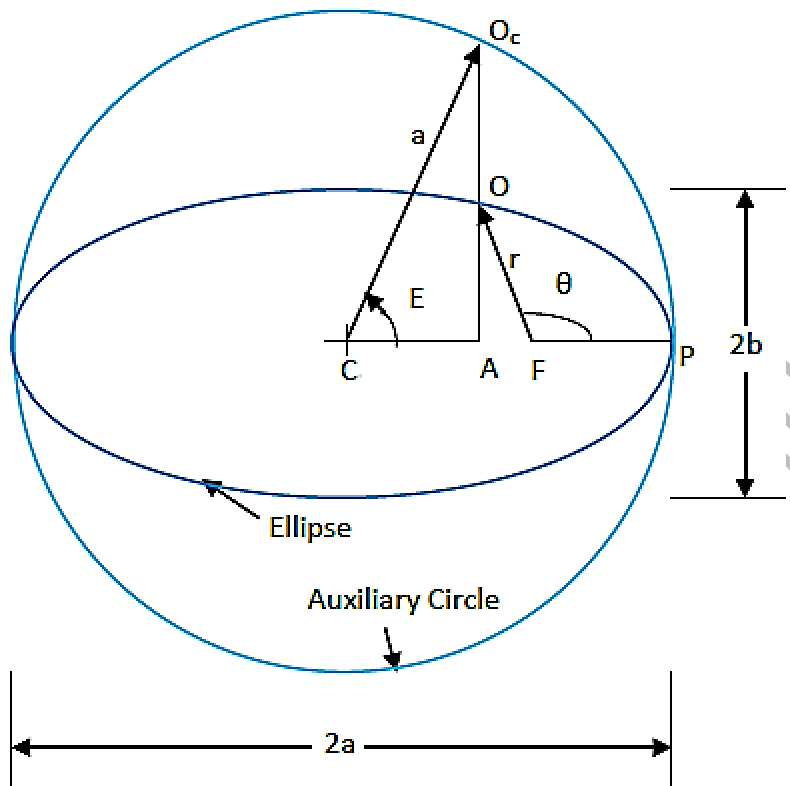


Figure 2: The Ellipse and its Auxiliary Circle

In figure 3 below, for an auxiliary circle,

$$Area\ FO_cP = A_c = Area_{sector}\ O_cCP - Area_{triangle}\ CO_cF$$

$$= \frac{1}{2}a^2E - \frac{1}{2} \times CF \times AO_c$$

$$= \frac{1}{2}a^2E - \frac{1}{2}\ .ae.a\sin E$$

$$= \frac{1}{2}\ a^2(E - e\sin E)$$

Now, since the Area OFP swept in the ellipse = $A_e = \frac{b}{a}A_c$

$$A_e = \int_0^{\theta} r^2 = \frac{h}{2}(t - t_0)$$

$$\frac{h}{2}(t - t_0) = \frac{ab}{2}[E - e\sin E]$$

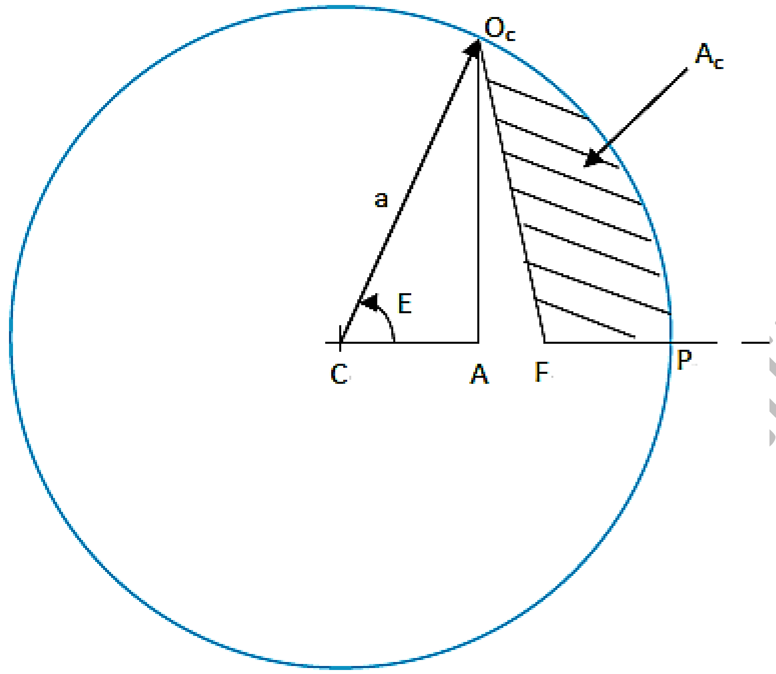


Figure 3

Recall

$$\frac{dA}{dt} = \frac{h}{2} = \frac{ab}{2}\sqrt{\frac{\mu}{a^3}} \qquad and \qquad T = 2\pi\sqrt{\frac{a^3}{\mu}} \; or \; \frac{2\pi}{t} = \sqrt{\frac{\mu}{a^3}} = \omega_m$$

Using the above,

$$\omega_m(t - t_0) = \sqrt{\frac{\mu}{a^3}}(t - t_0) = E - e\sin E = M_A$$

Here, ω_m is the angular frequency of the orbit. M_A is called the mean anomaly. The above equation is called the Kepler's Equation.

Now, we need to relate eccentric anomaly 'E' to the true anomaly 'θ'.

In Fig. 1, for triangles O_cCA and OAF,

$CA + AF = CF = ae$

$$=> \qquad a\cos E + r\cos(\pi - \theta) = ae$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

Therefore,

$$a\cos E = ae + \frac{a(1-e^2)\cos\theta}{1+e\cos\theta}$$

$$\boxed{\cos E = \frac{\cos\theta + e}{1+e\cos\theta}}$$

Similarly,

$$O_c A\left(\frac{b}{a}\right) = OA$$

$$a\sin E\frac{b}{a} = r\sin\theta \qquad and \qquad b = a\sqrt{1-e^2}$$

Therefore,

$$\boxed{\sin E = \frac{(1-e^2)\sin\theta}{1+e\cos\theta}}$$

For θ varying between 0 and π, E varies from 0 to π. However, the quadrants may be different in both cases. Similarly, for $\pi \le \theta \le 2\pi$, E also varies between π and 2π but the quadrants may be different. It is simpler to evaluate E as a tangent of half-angles where

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\left(\frac{E}{2}\right)}$$

The typical procedure for finding the Time of Flight (ToF) from periapsis to any point 'A' with known 'a', 'e' and 'θ' will involve

Determination of Eccentric anomaly (*E*) from true anomaly (θ) and *e*

Calculation of mean anomaly 'M_A'

Then the time of flight, $t = t_0 + \frac{M_A}{\omega_M}$

Example:

Given:
e = 0.9, r_p @ 600 km, μ_E = 4.0E14 m^3/s^2 and R_E = 6400 km, Calculate

ToF to 'A' @ θ = 135°

ToF 'A' to 'B' @ θ = 225°

Solution:

$r_P = a(1 - e) = 7000\ km$

This gives a = 70,000 km.

$$T = 2\pi\sqrt{\frac{a^3}{\mu_E}} = 2\pi\sqrt{\frac{70000 \times 10^3}{4 \times 10^{14}}} = 183991.15\ s \approx 51\ hrs$$

$$\omega_M = \sqrt{\frac{\mu_E}{a^3}} = \frac{1}{29283}\ rad/s$$

$$\tan\left(\frac{135}{2}\right) = \sqrt{\left(\frac{1+0.9}{1-0.9}\right)}\tan\left(\frac{E}{2}\right)$$

This gives E = 57.96° = 1.011 rad

Now, $M_A = 1.011 - 0.9\sin(57.96°) = 0.25\ rad$

Thus,

$$\tan\left(\frac{225}{2}\right) = \sqrt{\left(\frac{1+0.9}{1-0.9}\right)}\tan\left(\frac{E}{2}\right)$$

This gives E = - 57.96^o = - 1.011 rad (Note: This lies in the 4th quadrant!)

or E = 360^o – 57.96^o = 302.4^o = 5.27 rad

MB = 5.27 – 0.9 sin (302.4^o) = 6.03

Thus,

$$t_B - t_0 = \frac{M_B}{\omega_M} = 6.03 \times 29283 = 176576\, s \approx 49\, hrs$$

The difference is t_B–t_A = 47 hours.

REMAINING PART OF THE NOTES WILL BE PUBLISHED SOON